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# Mathematical Thinking: Studying the Notion of 'Transfer'

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Abstract: *These analyses form part of a three-year project[*] looking at mathematical thinking as a socially organised activity. We revisit data from a University Calculus class using tools from two theoretical perspectives, used increasingly in mathematics education research: (1) semiotic mediation and (2) discursive practices. We highlight how different theory-driven analyses taking a sociocultural view of thinking and learning can offer insights into the conceptualisation of the 'transfer' of learning.*

## Recent research on transfer

Mathematics educators have always maintained that the mathematical knowledge that students acquire in school should be able to be adapted and applied in workplace and everyday situations. It has also been recognised by the community that, far too often, it does not happen. Theoretical explanations for this failure depend on the author's views on the nature of the boundaries between the practices involved (see Muller & Taylor, 1995; Evans, 1999). Three main positions can be identified in the literature:

(1) The boundary between the everyday and school mathematics as permeable and theoretically unproblematic, if practically a considerable challenge for pedagogy. Inadequate instruction or inadequate learning can result in instrumental understanding (Skemp, 1976). Here, some authors emphasise the value to students' learning of the use of *authentic* contexts (Sullivan, Warren & White, 1999).

(2) Transfer is not possible, because of the impermeable boundaries between contexts and practices. These include the strongly situated view (often drawing on Lave (1988)) that meanings are produced and remain within specific social and cultural practices.

(3) Transfer is problematical, since boundaries exist, and are not automatically crossed, but still it is possible to enable something like 'transfer'. This includes the use of concepts such as 'consequential transitions' (Beach, 1999), 'translation' (Evans, 1999, 2000), and 'recontextualisation' (Cooper & Dunne, 1999). Amongst those who consider the boundary as problematic, Boaler (1998) argues that developing identities in communities of practice in which "students are enculturated and apprenticed into a

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system of knowing, thinking and doing" (Boaler, 1998, p. 118) might be helpful in enabling students to transfer their knowledge (see also Goodchild, 1999).

In our larger project, we have focused on three key issues – emotion (Matos & Evans, 2001), assessment (Morgan & Lerman, 2000) and transfer – as aspects of mathematical thinking, and we have deployed a range of theoretical perspectives to look at mathematical thinking as *socially organised*. We have taken data produced for other purposes and developed tools from those perspectives in order to read the data for when and how transfer occurs *as interpreted through those theoretical lenses*.

## The context of the production of the data

In this case, the data were collected during the second semester of 1996 in a First Calculus Course for university students taking a Business degree. The work reported was developed at the University of Algarve, as part of a research project aimed at studying the process of meaning making in mathematics learning and, particularly, the influence of real world situations in the meanings produced (Carreira, 1998).

From a methodological point of view, the research design assumed the double character of teaching experiment and curriculum development program. The teacher had very firm intentions of innovating and questioning the traditional format that such courses tend to assume. One premise that became central to the structure of the course was privileging the activity of mathematisation over the mastering of mathematical techniques and proficiency on specific topics (Moreira & Carreira, 1998).

On such a basis, the guiding lines of the curriculum emphasised: (1) the development of connections between mathematics and reality; (2) students' co-operative work on applied problem situations; (3) students' oral and written presentations of the work developed; (4) whole-class discussions on critical mathematical models and applications.

At the beginning of the session, students were given some information on a particular model describing the 'utility' of wine and beer. Their subsequent activity evolved around open questions that guided their exploration and investigation of the model.

> *We say that two commodities are competitive when the consumer tries not to exclude the consumption of neither of them. Assuming that the consumer's satisfaction can be measured in some way, economists have created the notion of utility to describe the degree of the consumer's satisfaction. In particular, this concept may be used to describe the utility of both wine and beer to a certain consumer.*
>
> *Consider the utility function $U(x,y)= (x+1)(y+2)$ as representing the utility that a certain consumer gets from the consumption of x units of wine and y units of beer.*

Previously, the teacher had explained to the class what level curves are, in the context of multivariable functions. Afterwards students were asked to interpret the designation of indifference curves used in Economics for contexts like the given utility model.

33. Cristina: I have heard of level curves in cartography and I worked a lot with that.
34. Isabel: But that kind of curve is only used in economics.

35. Cristina: Excuse me! In cartography there are also level curves! If you want to represent the terrain elevation on a system of axes, you get something like this... You can mark the heights above the sea... for instance 200 meters, 300 meters.

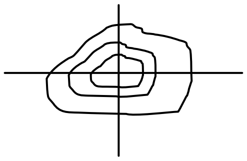

*Figure 1. Cristina's representation of level curves*

36. Isabel: No. In economics it’s different, it's like this...

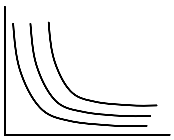

*Figure 2. Isabel's representation of level curves*

37. Cristina: In geography, level curves are usually defined with the help of a land survey, that is, a collection of data on the elevation of the land.
38. Eduardo: Those curves indicate different altitudes on a map.
39. Cristina: Well, in economics it must be something of the kind, I mean, in the same line of reasoning.
40. Isabel: No. What I think is that it must be more or less this... To have the same degree of satisfaction, we have to... follow one of these curves.
41. Cristina: What you're saying is that to get the same satisfaction, if the amount of beer increases, the amount of wine must decrease.
42. Isabel: Each of these curves is one utility...
43. Cristina: Yes, it shows that utility is always the same. And to have the utility unchanged, the variables have to change. If one increases, the other decreases.
44. Miguel: Each of the curves gives a different relationship between the variables x and y.
45. Cristina: Of course. It's like in geography. For instance, this utility is different from that one.
46. Miguel: But along each curve the utility doesn't change.
47. Cristina: It only changes from curve to curve.
48. Isabel: Each curve is a different level of utility.
49. Eduardo: But why are they called indifference curves?
50. Cristina: Because regardless of the values of the variables, the utility is indifferent.
51. Eduardo: I see. They're indifference curves because each curve has always the same level of utility. A curve is different only in relation to the other curves.
52. Cristina: The utility stays constant. Only the x and y change to make the utility unchanged.
53. Isabel: But what has that to do with cartography?
54. Cristina: It must have something. Look, I have this mountain. This inside area tells me the maximum height...
55. Eduardo: The point is that you have several levels. Except that here you have utility levels. Here the mountain is the utility. The consumers' satisfaction is also rising isn't it?

## A semiotic mediation analysis

From a semiotic mediation point of view, transfer is seen as a process that is not automatically an outcome of learning. On the contrary, the conditions of learning are given a decisive role in facilitating the occurrence of transfer. Transfer would be conceived as something that can and needs to be taught if one of the aims of teaching is to help students to make connections between different semiotic and conceptual systems. To allow for transfer may be thought of as creating the opportunities for students to engage with different conceptual tools, and to work with them simultaneously as happens in the case of metaphorical thinking (Carreira 1997, 1998). In addressing the data from this perspective, we will focus on the following aspects:

(a) students' verbal interactions as instances of the production of interpretants (that is, what makes the sign mean something to a particular individual, in a particular context) for mathematical signs

(b) students' mathematical thinking as instances of metaphorical thinking

<u>Chains of interpretants: The interweaving between mathematics and other conceptual domains.</u> A significant semiotic chain can be traced from the way students thought about the concept of level curve. At first, two of the students showed different conceptions of level curve, one coming from the field of cartography [33] and the other from economics [34]. They tended to see them as completely independent things: there seemed to be no clear connection between their contrasting sketches of level curves.

In spite of the apparent disconnection, students struggled to find some way of bridging the two conceptual domains. It was a shared effort which highlights an on-going *social* process of introducing successive interpretants based on the articulation of diverse semiotic means supported by student's differently experienced ideas. Cristina, for instance, tried to add information on the process underlying the depiction of level curves (contour lines) in cartography [35], [37]. She talks about altitude, about a land survey and explains how the different curves indicate different heights on a map. Other students recognise that in economics, level curves outline points for which the utility remains constant. They were also able to understand these curves as formal mathematical representations of a relationship between the independent variables [44], [52]. From the sketch produced by Isabel, we see that they observe the fact that an increase of one variable corresponds to a decrease of the other.

The overall process of interpreting the concept of level curve can be mapped as a sequence of interpretants, each of them tied to a certain referential domain:

- *Representation of terrain elevation*: "cartography", "land survey", "elevation of the land", "different altitudes on a map".

- *Path defining a certain constant utility*: "economics", "the same degree of satisfaction", "follow the curve", "along each curve the utility doesn't change", "each curve is one utility".

- *Representation of the relationship between x and y, for a given value of U*: "the variables have to change", "if one increases, the other decreases", "a different relationship between the variables x and y", "only the x and y change to make the utility unchanged".

Transfer: Models as the surface of conceptual metaphors. One aspect that is quite central in the data is that students insisted on finding some way of bridging two apparently separate conceptual fields. They wondered what the possible link might be and in their search they eventually came up with a metaphor to describe it – "the mountain is the utility" [55]. This is an outline of the implicit mapping contained in the metaphor:

In a mountain there are several height levels / Consumers experience different levels of satisfaction

The mountain rises up / Utility increases with the increase of consumption

To walk on a level curve of a mountain is to keep a certain altitude, in spite of the change of position / To follow a level curve of utility is to preserve a certain satisfaction, in spite of the change of the amounts of wine and beer.

The anchoring in two situations, which is visible in this metaphor, namely by tying utility, indifference, and consumption to mountains, heights, altitudes, reveals how students brought into their reasoning their specific knowledge and familiarity with different meaningful contexts. This takes us to the claim that transfer is closely related to the production of meanings in mathematics classrooms and that such meanings are not independent from the pedagogical scenario where learning takes place.

From a semiotic mediation point of view, mathematical models of real situations are important mediating tools in uncovering powerful underlying metaphors and in fostering metaphorical thinking. If students engage in exploring the multiple facets of a model, they have the opportunity to come up with genuine mathematical thinking in light of other cultural and semiotic systems. This is a mathematical practice and it can also be a school mathematical practice if models are to be seen as revealing something rather than conveying some frozen and fixed school mathematical content.

## A discursive practice analysis

From a discursive practice perspective, we conceptualise transfer as occurring when concepts originating in one discourse are linked to concepts of another discourse through a *chain of signification* (Evans, 2000). Such chains do not have an independent existence but arise for each participant (or, as in this case, for a group of participants) as they use their personal discursive resources, history and positionings to make meanings within a specific context (cf. Morgan, 1996). In addressing the data from this perspective, we need to identify the *resources* available to the participants and attempt to follow possible chains of signification through the developing conversation in the classroom. We also need to identify the available *positions* and their potentials for promoting or preventing shifts between discourses but, for reasons of space, we omit this part of the analysis.

The tasks for the analyst:

a) Identify the discourses available and the relevant concepts, values and relationships within these discourses.

b) In the transcript, look at the text as a whole and trace the discursive resources through the text. Identify whether there are key signifiers that play a role across discourses and attempt to follow the chains of signification. What is the contribution of each discourse to the solution of the problem and how are links between discourses constructed by and for the group as a whole?

<u>The discourses available:</u> The consumption of wine and beer may involve a number of ‘everyday’ discourses with which the students are familiar. At the same time, there are a number of ‘esoteric’ or academic discourses that are relevant to the problem as posed by the teacher and that the students may be able to draw on. The students are, of course, also participating in a classroom discourse with its own norms and available positions.

*Everyday discourses*: Consumption of wine and beer is generally located within particular patterns of social engagement – in various settings, with family or friends, at specific times. There is often strong regulation of the amount and type of consumption in a particular setting and the sort of behaviour that may accompany it and the meanings of consumption may vary considerably from one everyday practice to another. The everyday consumption of beer and wine may also involve a type of economic discourse. Unlike the esoteric economic discourse, however, this generally involves a *local* calculation – “Shall I buy beer or wine today, given my circumstances (my needs, desires, finances) today or this week?”

*Esoteric discourses*: The problem posed for the students and their identities in the context of this university class as students of Business Studies and of mathematics highlights certain discourses of academic subjects. In particular, resources from both economics and mathematics are to be seen in the form of the problem itself. As we see in the transcript, the students may also be able to draw on resources from other areas of their academic experience. The discourse of academic economics objectifies the experience of individuals and incorporates it into a *global* calculation, using terms such as ‘indifference’ that have meanings different from those associated with their use in everyday discourse. In everyday discourse the individual consumer is not indifferent to the specific make-up of a basket of purchases but will have preferences based on a range of non-financial criteria. It is only when ‘the consumer’ is conceived as an abstract generalised agent that ‘indifference’ occurs. Drawing on mathematical discourse creates a further abstraction. It is no longer important that the formula or the curve on the graph represents ‘indifference’ or that the purchase of beer or wine is involved. What is important is the algebraic and graphical representation of the problem, the relationships between these and the mathematical techniques for solving the problem that are facilitated by these representations.

*Classroom discourses*: The discourse of this classroom lays explicit value on making connections and on co-operation and communication between students. However, these students are likely also to be familiar with more traditional pedagogic discourses in which different subjects are strongly insulated and where individual work is more highly valued than group co-operation.

Tracing discursive resources in the transcript: Several aspects of the transcript might be analysed to give insight into transfer processes from a discursive perspective, for example: the ways the different students engage (or not) with the geographical discourse; the chain of signification formed by the shifting use of the terms *same*, *different*, *change*, *indifferent* and *indifference*; relationships between each student's positioning within the class and their use of resources from particular discourses. In the limited space available in this paper, however, we will focus on a brief section of the transcript in which may be seen a move from use of resources from everyday discourse and economics discourse to use of resources from mathematical discourse [lines 40-44]. The key terms are abstracted in the table below, showing three chains of signification.

| | | | |
|---|---|---|---|
| [40 Isabel] | satisfaction (*everyday/economics*) | | |
| [41 Cristina] | satisfaction | increase/decrease (*everyday/economics*) | in beer and wine (*everyday*) |
| [42 Isabel] | utility (*economics*) | | |
| [43 Cristina] | utility | increase/decrease | in variables (*economics/mathematics*) |
| [44 Miguel] | | relationship (*mathematics*) | between variables $x$ and $y$ (*mathematics*) |

The move from *satisfaction* [40 & 41] to *utility* [42 & 43] is one that is provided by the wording of the given problem but this shift away from everyday to esoteric economic discourse is associated with a parallel move from the everyday discourse of beer and wine [41] to the esoteric mathematical discourse of variables [43]. A similar move is made from the everyday notion of *increase* and *decrease* [41 & 43] towards the mathematical *relationship* [44]. The final move is achieved by Miguel rather than by the two women who dominate the discussion, though Cristina, at least, seems to accept it as meaningful [45]. The students play different roles in the process of making links between the concepts arising in everyday, economics and mathematics discourses. Miguel, in particular, introduces the mathematical resources that are later picked up by Cristina [52]; this role seems compatible with other descriptions (Carreira, 1998) of his behaviour in the group but would need confirmation by further data. We see the group as a whole achieving success with each member contributing to the chains of signification leading from everyday concepts to mathematical ones. 'Transfer' arises as a product of the play among the resources each participant brings to the group interaction.

## Conclusions and some methodological remarks

Both of our analyses are located in position (3) on transfer (see above): we see it as problematical, but capable of being supported by the conditions of learning and the chains of signification set up. These two approaches pay detailed attention to the (choices of) words used by the participants, emphasising the role of language both as a resource for structuring the individual's participation in the social practice(s) and methodologically as the primary means by which we as researchers construct meanings for the practices we observe. Further both analyses presuppose the social organisation of learning and transfer, and in particular, depend on ideas of practices and discourses and the boundaries between them.

Following the semiotic mediation position, it is possible to understand the term 'transfer' as a metaphor (Beach, 1999) for the processes occurring, when 'anything like transfer' takes place. Such would be the case, too, for the production of metaphorical meaning here described as a 'double-anchored meaning', in the sense that it reflects a connection between two objects (mountain, utility) and two interpretants (height, level) that are primarily tied to distinctive semiotic chains. Our analyses show that these processes – all them 'translations' or 'transitions' - are much more complicated than traditional views (see (1) above) suspect. They also suggest that mathematics educators can be more hopeful than the strongly insulationist positions ((2) above) allow.

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