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On self-normalised sums. Supplement.

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Abstract

This supplement aims to correct a misprint and clarify the point about the accuracy of Berry–Esseen–type inequalities for self-normalised sums and Student’s statistic established in [1].

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Let X, X_1, X_2, \dots be a sequence of independent and identically distributed random variables. Denote $t_n = \sum_{i=1}^n X_i / \sqrt{\sum_{i=1}^n X_i^2}$ (here and in the sequel we use the notation from [1]).

Theorems 9 and 10 in [1] establish uniform and non-uniform Berry–Esseen–type inequalities for self-normalised sums t_n and Student’s statistic.

The lower and upper bounds for $\mathbb{P}(t_n < x) - \Phi(x)$ in [1] are given with explicit constants, and the only moment assumption is the finiteness of the 3rd moment. The upper bound is of order $n^{-1/2}$. The lower bound is of order $n^{-2/7}$; it is of order $n^{-1/2}$ if the 4th moment is finite. In particular,

$$\sup_x |\mathbb{P}(t_n < x) - \Phi(x)| \leq C_* \frac{\mathbb{E}|X|^3}{\sqrt{n}} + \frac{8|\mathbb{E}X|^3}{e\sqrt{2\pi n}} + \frac{\sqrt{\mathbb{D}X^2}}{\sqrt{n}} \left(\frac{2}{\pi e}\right)^{1/4} + o(n^{-1/2}), \quad (1)$$

where C_* be the constant from the uniform Berry–Esseen inequality (the term $8\mathbb{E}|X|^3/(e\sqrt{2\pi n})$ in [1], p. 424, may be replaced by $8|\mathbb{E}X|^3/(e\sqrt{2\pi n})$).

The author has received a feedback that the bounds of Theorems 9 and 10 are not easy to understand. One distinguished scientist wrote that (1) was the only inequality with explicit constants he was able to find in [1].

The aim of this letter is to clarify that point and correct a misprint. We show that Theorems 9 and 10 provide bounds with explicit constants. Moreover, the bounds are sharp in the following sense: for a class of probability distributions, the estimates are asymptotically as sharp as those of the uniform and non-uniform Berry–Esseen inequalities.

There is a misprint in the definition of $R(t, a, b)$ on page 423: “ c ” in the denominator must be erased. Examples below illustrate the sharpness of the Berry–Esseen–type inequalities for self–normalised sums.

Example 1. Let $\mathbb{P}(X = -1) = \mathbb{P}(X = 1) = 1/2$. Then Theorem 9 with $\varepsilon = 0$ and $N = 1$ yields

$$\Delta_n \equiv \sup_x |\mathbb{P}(t_n < x) - \Phi(x)| \leq C_*/\sqrt{n}. \quad (2)$$

The right–hand side of (2) coincides with that of the Berry–Esseen theorem.

Put $\varepsilon_{n,x} = 0$, and notice that

$$\rho_n = p_n = \delta_{n,x} = \gamma_n^+ = 0, \quad R_n^+(x, 3) = C_+ n^{-1/2},$$

where C_+ the constant from the non–uniform Berry–Esseen inequality. Theorem 10 yields

$$|\mathbb{P}(t_n < x) - \Phi(x)| \leq C_+ n^{-1/2} (1 + x^3)^{-1}. \quad (3)$$

The right–hand side of estimate (3) coincides with that of the non–uniform Berry–Esseen inequality.

Example 2. Let $\mathbb{P}(X = -N) = \mathbb{P}(X = N) = 1/(2N^2)$, $\mathbb{P}(X = 0) = 1 - N^{-2}$. Denote $R_{n,N} = \frac{N}{\sqrt{n}} [C_*^{1/3} + C_+^{1/3} N/\sqrt{n}]^3$ and $r_{n,N} = \frac{24\sqrt{6}N^2}{e\sqrt{\pi\varepsilon n}}$. It is easy to check that

$$\begin{aligned} \mathbb{E}X &= 0, \quad \mathbb{E}|X|^m = N^{m-2} \quad (m \in \mathbb{N}), \quad \rho_n = m_0 = 0, \\ \beta_n &= e^{-n/(36N^2)}, \quad \max\{R_n(3, 1, 1); R_n(3, 2, 2)\} \leq R_{n,N}, \\ r_n &< 2r_{n,N}, \quad \mathbb{P}\left(\sum_{i=1}^n (X_i^2 - 1)/n > \sqrt{2\varepsilon}\right) \leq e^{-\varepsilon n/(2N^2)}. \end{aligned}$$

Theorem 9 gives

$$\Phi(x(1 - \varepsilon)) - R_{n,N} - 2r_{n,N} - e^{-\varepsilon n/(2N^2)} \leq \mathbb{P}(t_n < x) \leq \Phi(x) + R_{n,N} + r_{n,N}. \quad (4)$$

Assume that $N = N(n) \rightarrow \infty$, $N^2/n \rightarrow 0$. With $\varepsilon = 2n^{-1}N^2 \ln(n/N^2)$, (4) implies

$$\Delta_n \leq C_* N n^{-1/2} + O\left(n^{-1} N^2 \ln\left(n/N^2\right)\right).$$

In other words, estimate (4) is asymptotically as sharp as that of the Berry–Esseen theorem.

In order to apply Theorem 10, note that $\gamma_n^+ \leq \gamma_{n,x}^+$ and $R_n^+(x, 3) \leq R_{n,x}^+$, where

$$\gamma_{n,x}^+ = 9x^2 N^2 / 25n, \quad R_{n,x}^+ = C_+ N n^{-1/2} \left(1 + 3xN/(5\sqrt{n})\right)^3.$$

Besides, $\mu_3^> = 0$ if $x \geq 6N/\sqrt{n}$. Theorem 10 yields

$$\begin{aligned} & \Phi\left(x(1-\varepsilon)(1-\gamma_{n,x}^+)\right) - R_{n,x}^+ \left(1+x^3(1-\gamma_{n,x}^+)^3/(1-\varepsilon)^3\right)^{-1} - e^{-\varepsilon n/(2N^2)} \\ & \leq \mathbb{P}(t_n < x) \leq \Phi(x) + R_{n,x}^+ \left(1+x^3(1-\gamma_{n,x}^+)^3\right)^{-1} + e^{-n/(36N^2)} \end{aligned} \quad (5)$$

as $6N/\sqrt{n} \leq x \leq \sqrt{n}/(3N)$. With $\varepsilon = 2n^{-1}N^2 \ln(nN^{-2})(2+x^2)$, inequality (5) implies

$$\begin{aligned} & -C_+ \left(1+3xN/(5\sqrt{n})\right)^3 - v_n n^{-1} N^2 \ln(n/N^2) \leq (1+x^3)[\mathbb{P}(t_n < x) - \Phi(x)]\sqrt{n}/N \\ & \leq C_+ \left(1+3xN/(5\sqrt{n})\right)^3 (1+x^3) \left(1+x^3(1-\gamma_{n,x}^+)\right)^{-1} + v_n n^{-1} N^2, \end{aligned}$$

where $v_n \rightarrow 0$ uniformly in $x \in [6N/\sqrt{n}; \sqrt{n}/(3N)]$. Let $\{u_n\}$ be a sequence of positive numbers such that $u_n \rightarrow 0$ and $u_n\sqrt{n}/N \rightarrow \infty$. Then

$$(1+x^3)|\mathbb{P}(t_n < x) - \Phi(x)|\sqrt{n}/N \leq C_+ + o(1) \quad (6)$$

uniformly in $x \in [0; u_n\sqrt{n}/N]$.

In the expanding interval $[0; u_n\sqrt{n}/N]$, estimate (6) is asymptotically as sharp as that of the non-uniform Berry–Esseen bound. We notice in [2] that a non-uniform Berry–Esseen-type inequality for self-normalised sums may not, in general, hold on the whole line.

References

- [1] S.Y. Novak (2000) On self-normalised sums. — *Math. Methods Statist.*, v. 9, No 4, 415–436.
- [2] S.Y. Novak (2001) On self-normalised sums and Student’s statistic. — Brunel University: Technical Report No TR/18/01.