QPTAS and Subexponential Algorithm for Maximum Clique on Disk Graphs

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Abstract

A (unit) disk graph is the intersection graph of closed (unit) disks in the plane. Almost three decades ago, an elegant polynomial-time algorithm was found for Maximum Clique on unit disk graphs [Clark, Colbourn, Johnson; Discrete Mathematics '90]. Since then, it has been an intriguing open question whether or not tractability can be extended to general disk graphs. We show the rather surprising structural result that a disjoint union of cycles is the complement of a disk graph if and only if at most one of those cycles is of odd length. From that, we derive the first QPTAS and subexponential algorithm running in time $2^{\tilde{O}(n^{2/3})}$ for Maximum Clique on disk graphs. In stark contrast, Maximum Clique on intersection graphs of filled ellipses or filled triangles is unlikely to have such algorithms, even when the ellipses are close to unit disks. Indeed, we show that there is a constant ratio of approximation which cannot be attained even in time $2^{n^{1-\varepsilon}}$, unless the Exponential Time Hypothesis fails.

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1 Introduction

An *intersection graph* of geometric objects has one vertex per object and an edge between every pair of vertices corresponding to intersecting objects. Intersection graphs for many

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different families of geometric objects have been studied due to their practical applications and rich structural properties [10, 31]. Among the most studied ones are *disk graphs*, which are intersection graphs of closed disks in the plane, and their special case, *unit disk graphs*, where all the radii are the same. Their applications range from sensor networks to map labeling [21], and many standard optimization problems have been studied on disk graphs, see for example [36] and references therein. In this paper, we study Maximum Clique on general disk graphs.

Known results.

Recognizing unit disk graphs is NP-hard [11], and even $\exists\mathbb{R}$-complete [25]. Clark et al. [19] gave a polynomial-time algorithm for Maximum Clique on unit disk graphs with a geometric representation. The core idea of their algorithm can actually be adapted so that the geometric representation is no longer needed [34]. The complexity of the problem on general disk graphs is unfortunately still unknown. Using the fact that the transversal number for disks is 4, Ambühl and Wagner [4] gave a simple 2-approximation algorithm for Maximum Clique on general disk graphs. They also showed the problem to be APX-hard on intersection graphs of ellipses and gave a $9\rho^2$-approximation algorithm for filled ellipses of aspect ratio at most ρ. Since then, the problem has proved to be elusive with no new positive or negative results. The question on the complexity and further approximability of Maximum Clique on general disk graphs is considered as folklore [6], but was also explicitly mentioned as an open problem by Fishkin [21], Ambühl and Wagner [4] and Cabello [13, 14].

A closely related problem is Maximum Independent Set, which is known to be W[1]-hard (even on unit disk graphs [30]) and to admit a subexponential exact algorithm [2] and PTAS [17, 20] on disk graphs.

Results and organization.

In Section 2, we mainly prove that the disjoint union of two odd cycles is not the complement of a disk graph. To the best of our knowledge, this is the first structural property that general disk graphs do not inherit from strings or from convex objects. We provide an infinite family of forbidden induced subgraphs, an analogue to the recent work of Atminas and Zamaraev on unit disk graphs [5]. In Section 3, we show how to use this structural result to approximate and solve Maximum Independent Set on complements of disk graphs, hence Maximum Clique on disk graphs. More precisely, we present the first quasi-polynomial-time approximation scheme (QPTAS) and subexponential-time algorithm for Maximum Clique on disk graphs, even without the geometric representation of the graph. In Section 4, we highlight how those algorithms contrast with the situation for ellipses or triangles, where there is a constant $\alpha > 1$ for which an α-approximation running in subexponential time is highly unlikely (in particular, ruling out at once QPTAS *and* subexponential-time algorithm). We conclude in Section 5 with a few open questions.

Definitions and notations.

For two integers $i \leqslant j$, we denote by $[i, j]$ the set of integers $\{i, i+1, \ldots, j-1, j\}$. For a positive integer i, we denote by $[i]$ the set of integers $[1, i]$. If S is a subset of vertices of a graph, we denote by $N(S)$ the open neighborhood of S and by $N[S]$ the set $N(S) \cup S$. The *2-subdivision* of a graph G is the graph H obtained by subdividing each edge of G exactly twice. If G has n vertices and m edges, then H has $n + 2m$ vertices and $3m$ edges. The *co-2-subdivision* of G is the complement of H. Hence it has $n+2m$ vertices and $\binom{n+2m}{2} - 3m$

edges. The *co-degree* of a graph is the maximum degree of its complement. A *co-disk* is a graph that is the complement of a disk graph.

For two distinct points x and y in the plane, we denote by $\ell(x, y)$ the unique line going through x and y, and by $\text{seg}(x, y)$ the closed straight-line segment whose endpoints are x and y. If s is a segment with positive length, then we denote by $\ell(s)$ the unique line containing s. We denote by $d(x, y)$ the euclidean distance between points x and y. We will often define disks and elliptical disks by their boundary, i.e., circles and ellipses, and also use the following basic facts. There are exactly two circles that pass through a given point with a given tangent at this point and a given radius; one if we further specify on which side of the tangent the circle is. There is exactly one circle which passes through two points with a given tangent at one of the two points, provided the other point is *not* on this tangent. Finally, there exists one (not necessarily unique) ellipse which passes through two given points with two given tangents at those points.

The *Exponential Time Hypothesis* (ETH) is a conjecture by Impagliazzo et al. asserting that there is no $2^{o(n)}$-time algorithm for 3-SAT on instances with n variables [24]. The ETH, together with the sparsification lemma [24], even implies that there is no $2^{o(n+m)}$-time algorithm solving 3-SAT.

2 Disk graphs with co-degree 2

In this section, we fully characterize the degree-2 complements of disk graphs. We show the following:

▶ **Theorem 1.** *A disjoint union of paths and cycles is the complement of a disk graph if and only if the number of odd cycles is at most one.*

We split this theorem into two parts. In the first one, Section 2.1, we show that the union of two disjoint odd cycles is not the complement of a disk graph. This is the part that will be algorithmically useful. As disk graphs are closed under taking induced subgraphs, it implies that in the complement of a disk graph two vertex-disjoint odd cycles have to be linked by at least one edge. This will turn out useful when solving Maximum Independent Set on the complement of the graph (to solve Maximum Clique on the original graph). In the second part, Section 2.2, we show how to represent the complement of the disjoint union of even cycles and exactly one odd cycle. Although this result is not needed for the forthcoming algorithmic section, it nicely highlights the singular role that parity plays and exposes the complete set of disk graphs of co-degree 2.

2.1 The disjoint union of two odd cycles is not co-disk

We call *positive distance* between two non-intersecting disks the minimum of $d(x, y)$ where x is in one disk and y is in the other. If the disks are centered at c_1 and c_2 with radius r_1 and r_2, respectively, then this value is $d(c_1, c_2) - r_1 - r_2$. We call *negative distance* between two intersecting disks the length of the straight-line segment defined as the intersection of three objects: the two disks and the line joining their center. This value is $r_1 + r_2 - d(c_1, c_2)$, which is positive.

We call *proper representation* a disk representation where every edge is witnessed by a proper intersection of the two corresponding disks, i.e., the interiors of the two disks intersect. It is easy to transform a disk representation into a proper representation (of the same graph).

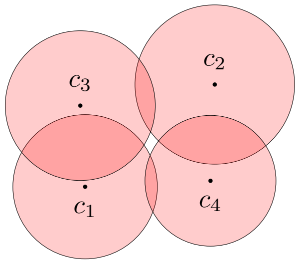

Figure 1 Disk realization of a $K_{2,2}$. As the centers are positioned, it is impossible that the two non-edges are between the disks 2 and 3, and between the disks 1 and 4 (or between the disks 1 and 3, and between the disks 2 and 4).

▶ **Lemma 2.** *If a graph has a disk representation, then it has a proper representation.*

Proof. If two disks intersect non-properly, we increase the radius of one of them by $\varepsilon/2$ where ε is the smallest positive distance between two disks. ◀

In order not to have to discuss about the corner case of three aligned centers in a disk representation, we show that such a configuration is never needed to represent a disk graph.

▶ **Lemma 3.** *If a graph has a disk representation, it has a proper representation where no three centers are aligned.*

Proof. By Lemma 2, we have or obtain a proper representation. Let ε be the minimum between the smallest positive distance and the smallest negative distance. As the representation is proper, $\varepsilon > 0$. If three centers are aligned, we move one of them to any point which is not lying in a line defined by two centers in a ball of radius $\varepsilon/2$ centered at it. This decreases by at least one the number of triple of aligned centers, and can be repeated until no three centers are aligned. ◀

From now on, we assume that every disk representation is proper and without three aligned centers. We show the folklore result that in a representation of a $K_{2,2}$ that sets the four centers in convex position, both non-edges have to be *diagonal*.

▶ **Lemma 4.** *In a disk representation of $K_{2,2}$ with the four centers in convex position, the non-edges are between vertices corresponding to opposite centers in the quadrangle.*

Proof. Let c_1 and c_2 be the centers of one non-edge, and c_3 and c_4 the centers of the other non-edge. Let r_i be the radius associated to center c_i for $i \in [4]$. It should be that $d(c_1, c_2) > r_1 + r_2$ and $d(c_3, c_4) > r_3 + r_4$ (see Figure 1). Assume c_1 and c_2 are consecutive on the convex hull formed by $\{c_1, c_2, c_3, c_4\}$, and say, without loss of generality, that the order is c_1, c_2, c_3, c_4. Let c be the intersection of $\text{seg}(c_1, c_3)$ and $\text{seg}(c_2, c_4)$. It holds that $d(c_1, c_3) + d(c_2, c_4) = d(c_1, c) + d(c, c_3) + d(c_2, c) + d(c, c_4) = (d(c_1, c) + d(c, c_2)) + (d(c_3, c) + d(c, c_4)) > d(c_1, c_2) + d(c_3, c_4) > r_1 + r_2 + r_3 + r_4 = (r_1 + r_3) + (r_2 + r_4)$. Which implies that $d(c_1, c_3) > r_1 + r_3$ or $d(c_2, c_4) > r_2 + r_4$; a contradiction. ◀

We derive a useful consequence of the previous lemma, phrased in terms of intersections of lines and segments.

▶ **Corollary 5.** *In any disk representation of $K_{2,2}$ with centers c_1, c_2, c_3, c_4 with the two non-edges between the vertices corresponding to c_1 and c_2, and between c_3 and c_4, it should be that $\ell(c_1, c_2)$ intersects $seg(c_3, c_4)$ or $\ell(c_3, c_4)$ intersects $seg(c_1, c_2)$.*

Proof. Either the disk representation has the four centers in convex position. In that case, by Lemma 4, $\text{seg}(c_1, c_2)$ and $\text{seg}(c_3, c_4)$ are the diagonals of a convex quadrangle. Hence they intersect, and *a fortiori*, $\ell(c_1, c_2)$ intersects $\text{seg}(c_3, c_4)$ ($\ell(c_3, c_4)$ intersects $\text{seg}(c_1, c_2)$, too).

Or the disk representation has one center, say without loss of generality, c_1, in the interior of the triangle formed by the other three centers. In that case, $\ell(c_1, c_2)$ intersects $\text{seg}(c_3, c_4)$. If instead a center in $\{c_3, c_4\}$ is in the interior of the triangle formed by the other centers, then $\ell(c_3, c_4)$ intersects $\text{seg}(c_1, c_2)$. ◀

We can now prove the main result of this section thanks to the previous corollary, parity arguments, and *some elementary properties of closed plane curves*, namely Property I and Property III of the eponymous paper [35].

▶ **Theorem 6.** *The complement of the disjoint union of two odd cycles is not a disk graph.*

Proof. Let s and t be two positive integers and $G = \overline{C_{2s+1} + C_{2t+1}}$ the complement of the disjoint union of a cycle of length $2s + 1$ and a cycle of length $2t + 1$. Assume that G is a disk graph. Let $\mathcal{C}_1$ (resp. $\mathcal{C}_2$) be the cycle embedded in the plane formed by $2s + 1$ (resp. $2t + 1$) straight-line segments joining the consecutive centers of disks along the first (resp. second) cycle. Observe that the segments of those two cycles correspond to the non-edges of G. We number the segments of $\mathcal{C}_1$ from S_1 to S_{2s+1}, and the segments of $\mathcal{C}_2$, from S'_1 to S'_{2t+1}.

For the i-th segment S_i of $\mathcal{C}_1$, let a_i be the number of segments of $\mathcal{C}_2$ intersected by the line $\ell(S_i)$ prolonging S_i, let b_i be the number of segments S'_j of $\mathcal{C}_2$ such that the prolonging line $\ell(S'_j)$ intersects S_i, and let c_i be the number of segments of $\mathcal{C}_2$ intersecting S_i. For the second cycle, we define similarly a'_j, b'_j, c'_j. The quantity $a_i + b_i - c_i$ counts the number of segments of $\mathcal{C}_2$ which can possibly represent a $K_{2,2}$ with S_i according to Corollary 5. As we assumed that G is a disk graph, $a_i + b_i - c_i = 2t + 1$ for every $i \in [2s + 1]$. Otherwise there would be at least one segment S'_j of $\mathcal{C}_2$ such that $\ell(S_i)$ does not intersect S'_j *and* $\ell(S'_j)$ does not intersect S_i.

Observe that a_i is an even integer since $\mathcal{C}_2$ is a closed curve. Also, $\Sigma_{i=1}^{2s+1} a_i + b_i - c_i = (2t + 1)(2s + 1)$ is an odd number, as the product of two odd numbers. This implies that $\Sigma_{i=1}^{2s+1} b_i - c_i$ shall be odd. $\Sigma_{i=1}^{2s+1} c_i$ counts the number of intersections of the two closed curves $\mathcal{C}_1$ and $\mathcal{C}_2$, and is therefore even. Hence, $\Sigma_{i=1}^{2s+1} b_i$ shall be odd. Observe that $\Sigma_{i=1}^{2s+1} b_i = \Sigma_{j=1}^{2t+1} a'_j$ by reordering and reinterpreting the sum from the point of view of the segments of $\mathcal{C}_2$. Since the a'_j are all even, $\Sigma_{i=1}^{2s+1} b_i$ is also even; a contradiction. ◀

2.2 The disjoint union of cycles with at most one odd is co-disk

We only show the following part of Theorem 1 to emphasize that, rather unexpectedly, parity plays a crucial role in disk graphs of co-degree 2. It is also amusing that the complement of any odd cycle is a *unit* disk graph while the complement of any even cycle of length at least 8 is not [5]. Here, the situation is somewhat reversed: complements of even cycles are *easier* to represent than complements of odd cycles. The proof of the following theorem can be found in the full version [9].

▶ **Theorem 7.** *The complement of the disjoint union of even cycles and one odd cycle is a disk graph.*

Theorem 6 and Theorem 7, together with the fact that disk graphs are closed by taking induced subgraphs prove Theorem 1.

3 Algorithmic consequences

Now we show how to use the structural results from Section 2 to obtain algorithms for Maximum Clique in disk graphs. A clique in a graph G is an independent set in $\overline{G}$. So, leveraging the result from Theorem 1, we will focus on solving Maximum Independent Set in graphs without two vertex-disjoint odd cycles as an induced subgraph.

3.1 QPTAS

The odd cycle packing number $\mathrm{ocp}(H)$ of a graph H is the maximum number of vertex-disjoint odd cycles in H. Unfortunately, the condition that $\overline{G}$ does not contain two vertex-disjoint odd cycles as an induced subgraph is not quite the same as saying that the odd cycle packing number of $\overline{G}$ is 1. Otherwise, we would immediately get a PTAS by the following result of Bock et al. [7].

▶ **Theorem 8** (Bock et al. [7]). *For every fixed $\varepsilon > 0$ there is a polynomial $(1+\varepsilon)$-approximation algorithm for* Maximum Independent Set *for graphs H with n vertices and $\mathrm{ocp}(H) = o(n/\log n)$.*

The algorithm by Bock et al. works in polynomial time if $\mathrm{ocp}(H) = o(n/\log n)$, but it does not need the odd cycle packing explicitly given as an input. This is important, since finding a maximum odd cycle packing is NP-hard [26]. We start by proving a structural lemma, which spares us having to determine the odd cycle packing number.

▶ **Lemma 9.** *Let H be a graph with n vertices, whose complement is a disk graph. If $\mathrm{ocp}(H) > n/\log^2 n$, then H has a vertex of degree at least $n/\log^4 n$.*

Proof. Consider a maximum odd cycle packing $\mathcal{C}$. By assumption, it contains more than $n/\log^2 n$ vertex-disjoint cycles. By the pigeonhole principle, there must be a cycle $C \in \mathcal{C}$ of size at most $\log^2 n$. Now, by Theorem 6, H has no two vertex-disjoint odd cycles with no edges between them. Therefore there must be an edge from C to every other cycle of $\mathcal{C}$, there are at least $n/\log^2 n$ such edges. Let v be a vertex of C with the maximum number of edges to other cycles in $\mathcal{C}$, by the pigeonhole principle its degree is at least $n/\log^4 n$. ◀

Now we are ready to construct a QPTAS for Maximum Clique in disk graphs.

▶ **Theorem 10.** *For any $\varepsilon > 0$,* Maximum Clique *can be $(1+\varepsilon)$-approximated in time $2^{O(\log^5 n)}$, when the input is a disk graph with n vertices.*

Proof. Let G be the input disk graph and let $\overline{G}$ be its complement, we want to find a $(1+\varepsilon)$-approximation for Maximum Independent Set in $\overline{G}$. We consider two cases. If $\overline{G}$ has no vertex of degree at least $n/\log^4 n$, then, by Lemma 9, we know that $\mathrm{ocp}(\overline{G}) \leqslant n/\log^2 n = o(n/\log n)$. In this case we run the PTAS of Bock et al. and we are done.

In the other case, $\overline{G}$ has a vertex v of degree at least $n/\log^4 n$ (note that it may still be the case that $\mathrm{ocp}(\overline{G}) = o(n/\log n)$). We branch on v: either we include v in our solution and remove it and all its neighbors, or we discard v. The complexity of this step is described by the recursion $F(n) \leqslant F(n-1) + F(n - n/\log^4 n)$ and solving it gives us the desired running time. Note that this step is exact, i.e., we do not lose any solutions. ◀

3.2 Subexponential algorithm

Now we will show how our structural result can be used to construct a subexponential algorithm for Maximum Clique in disk graphs. The *odd girth* of a graph is the size of a shortest odd cycle. An *odd cycle cover* is a subset of vertices whose deletion makes the graph bipartite. We will use a result by Györi et al. [23], which says that graphs with large odd girth have small odd cycle cover. In that sense, it can be seen as relativizing the fact that odd cycles do not have the Erdős-Pósa property. Bock et al. [7] turned the non-constructive proof into a polynomial-time algorithm.

▶ **Theorem 11** (Györi et al. [23], Bock et al. [7]). *Let H be a graph with n vertices and no odd cycle shorter than δn (δ may be a function of n). Then there is an odd cycle cover X of size at most $(48/\delta)\ln(5/\delta)$ Moreover, X can be found in polynomial time.*

Let us start with showing three variants of an algorithm.

▶ **Theorem 12.** *Let G be a disk graph with n vertices. Let Δ be the maximum degree of $\overline{G}$ and c the odd girth of $\overline{G}$ (they may be functions of n).* Maximum Clique *has a branching or can be solved, up to a polynomial factor, in time:*
(i) $2^{\tilde{O}(n/\Delta)}$ *(branching),* *(ii)* $2^{\tilde{O}(n/c)}$ *(solved),* *(iii)* $2^{O(c\Delta)}$ *(solved).*

Proof. Let G be the input disk graph and let $\overline{G}$ be its complement, we look for a maximum independent set in $\overline{G}$.

To prove (i), consider a vertex v of degree Δ in $\overline{G}$. We branch on v: either we include v in our solution and remove $N[v]$, or discard v. The complexity is described by the recursion $F(n) \leqslant F(n-1) + F(n-(\Delta+1))$ and solving it gives (i). Observe that this does not give an algorithm running in time $2^{\tilde{O}(n/\Delta)}$ since the maximum degree might drop. Therefore, we will do this branching as long as it is *good enough* and then finish with the algorithms corresponding to (ii) and (iii).

For (ii) and (iii), let C be the cycle of length c, it clearly can be found in polynomial time. By application of Theorem 11 with $\delta = c/n$, we find an odd cycle cover X in $\overline{G}$ of size $\tilde{O}(n/c)$ in polynomial time (see for instance [3]). Next we exhaustively guess in time $2^{\tilde{O}(n/c)}$ the intersection I of an optimum solution with X and finish by finding a maximum independent set in the bipartite graph $\overline{G} - (X \cup N(I))$, which can be done in polynomial time. The total complexity of this case is $2^{\tilde{O}(n/c)}$, which shows (ii).

Finally, observe that the graph $\overline{G} - N[C]$ is bipartite, since otherwise $\overline{G}$ contains two vertex-disjoint odd cycles with no edges between them. Moreover, since every vertex in $\overline{G}$ has degree at most Δ, it holds that $|N[C]| \leqslant c(\Delta-1) \leqslant c\Delta$. Indeed, a vertex of C can only have $c(\Delta-2)$ neighbors outside C. We can proceed as in the previous step: we exhaustively guess the intersection of the optimal solution with $N[C]$ and finish by finding the maximum independent set in a bipartite graph (a subgraph of $\overline{G} - N[C]$), which can be done in total time $2^{O(c\Delta)}$, which shows (iii). ◀

Now we show how the structure of G affect the bounds in Theorem 12.

▶ **Corollary 13.** *Let G be a disk graph with n vertices.* Maximum Clique *can be solved in time:*
(a) $2^{\tilde{O}(n^{2/3})}$*,*
(b) $2^{\tilde{O}(\sqrt{n})}$ *if the maximum degree of $\overline{G}$ is constant,*
(c) polynomial, if both the maximum degree and the odd girth of $\overline{G}$ are constant.

Proof. Δ and c can be computed in polynomial time. Therefore, knowing what is faster among cases (i), (ii), and (iii) is tractable. For case (a), while there is a vertex of degree at least $n^{1/3}$, we branch on it. When this process stops, we do what is more advantageous between cases (ii) and (iii). Note that $\min(n/\Delta, n/c, c\Delta) \leqslant n^{2/3}$ (the equality is met for $\Delta = c = n^{1/3}$). For case (b), we do what is best between cases (ii) and (iii). Note that $\min(n/c, c) \leqslant \sqrt{n}$ (the equality is met for $c = \sqrt{n}$). Finally, case (c) follows directly from case (iii) in Theorem 12. ◀

Observe that case (b) is typically the hardest one for Maximum Clique. Moreover, the win-win strategy of Corollary 13 can be directly applied to solve Maximum Weighted Clique, as finding a maximum weighted independent set in a bipartite graph is still polynomial-time solvable. On the other hand, this approach cannot be easily adapted to obtain a subexponential algorithm for Clique Partition (even Clique p-Partition with constant p), since List Coloring (even List 3-Coloring) has no subexponential algorithm for bipartite graphs, unless the ETH fails (see [28], the bound can be obtained if we start reduction from a sparse instance of 1-in-3-Sat instead of Planar 1-in-3-Sat).

4 Other intersection graphs and limits

In this section, we discuss the impossibility of generalizing our results to related classes of intersection graphs.

4.1 Filled ellipses and filled triangles

A natural generalization of a disk is an *elliptical disk*, also called *filled ellipse*, i.e., an ellipse plus its interior. The simplest convex set with non empty interior is a filled triangle (a triangle plus its interior). We show that our approach developed in the two previous sections, and actually every approach, is bound to fail for filled ellipses and filled triangles.

APX-hardness was shown for Maximum Clique in the intersection graphs of (*non-filled*) ellipses and triangles by Ambühl and Wagner [4]. Their reduction also implies that there is no subexponential algorithm for this problem, unless the ETH fails. Moreover, they claim that their hardness result extends to filled ellipses since *"intersection graphs of ellipses without interior are also intersection graphs of filled ellipses"*. Unfortunately, this claim is incorrect. In the full version, we show:

► **Theorem 14.** *There is a graph G which has an intersection representation with ellipses without their interior, but has no intersection representation with convex sets.*

This error and the confusion between filled ellipses and ellipses without their interior has propagated to other more recent papers [27]. Fortunately, we show that the hardness result does hold for filled ellipses (and filled triangles) with a different reduction. Our construction can be seen as streamlining the ideas of Ambühl and Wagner [4]. It is simpler and, in the case of (filled) ellipses, yields a somewhat stronger statement.

► **Theorem 15.** *There is a constant $\alpha > 1$ such that for every $\varepsilon > 0$,* Maximum Clique *on the intersection graphs of filled ellipses has no α-approximation algorithm running in subexponential time $2^{n^{1-\varepsilon}}$, unless the ETH fails, even when the ellipses have arbitrarily small eccentricity and arbitrarily close value of major axis.*

This is in sharp contrast with our subexponential algorithm and with our QPTAS when the eccentricity is 0 (case of disks). For any $\varepsilon > 0$, if the eccentricity is only allowed to be at most ε, a subexponential algorithm or a QPTAS are very unlikely. This result subsumes [16] (where NP-hardness is shown for connected shapes contained in a disk of radius 1 and containing a concentric disk of radius $1-\varepsilon$ for arbitrarily small $\varepsilon > 0$) and corrects [4]. We show the same hardness for the intersection graphs of filled triangles.

▶ **Theorem 16.** *There is a constant $\alpha > 1$ such that for every $\varepsilon > 0$,* Maximum Clique *on the intersection graphs of filled triangles has no α-approximation algorithm running in subexponential time $2^{n^{1-\varepsilon}}$, unless the ETH fails.*

We first show this lower bound for Maximum Weighted Independent Set on the class of all the 2-subdivisions, hence the same hardness for Maximum Weighted Clique on all the co-2-subdivisions. It is folklore that from the PCP of Moshkovitz and Raz [33], which roughly implies that Max 3-SAT cannot be $7/8+\varepsilon$-approximated in subexponential time under the ETH, one can derive such inapproximability in subexponential time for many hard graph and hypergraph problems; see for instance [8].

The following inapproximability result for Maximum Independent Set on bounded-degree graphs was shown by Chlebík and Chlebíková [18]. As their reduction is almost linear, the PCP of Moshkovitz and Raz boosts this hardness result from ruling out polynomial-time up to ruling out subexponential time $2^{n^{1-\varepsilon}}$ for any $\varepsilon > 0$.

▶ **Theorem 17** ([18, 33])**.** *There is a constant $\beta > 0$ such that* Maximum Independent Set *on graphs with n vertices and maximum degree Δ cannot be $1+\beta$-approximated in time $2^{n^{1-\varepsilon}}$ for any $\varepsilon > 0$, unless the ETH fails.*

We could actually state a slightly stronger statement for the running time but will settle for this for the sake of clarity. In the full version [9], we show the following:

▶ **Theorem 18.** *There is a constant $\alpha > 1$ such that for any $\varepsilon > 0$,* Maximum Independent Set *on the class of all the 2-subdivisions has no α-approximation algorithm running in subexponential time $2^{n^{1-\varepsilon}}$, unless the ETH fails.*

Proof. Let G be a graph with maximum degree a constant Δ, with n vertices $v_1, \ldots, v_n$ and m edges $e_1, \ldots, e_m$, and let H be its 2-subdivision. Recall that to form H, we subdivided every edge of G exactly twice. These $2m$ vertices in $V(H) \setminus V(G)$, representing edges, are called *edge vertices* and are denoted by $v^+(e_1), v^-(e_1), \ldots, v^+(e_m), v^-(e_m)$, as opposed to the other vertices of H, which we call *original vertices*. If $e_k = v_iv_j$ is an edge of G, then $v^+(e_k)$ (resp. $v^-(e_k)$) has two neighbors: $v^-(e_k)$ and v_i (resp. $v^+(e_k)$ and v_j).

Observe that there is a maximum independent set S which contains exactly one of $v^+(e_k), v^-(e_k)$ for every $k \in [m]$. Indeed, S cannot contain both $v^+(e_k)$ and $v^-(e_k)$ since they are adjacent. On the other hand, if S contains neither $v^+(e_k)$ nor $v^-(e_k)$, then adding $v^+(e_k)$ to S and potentially removing the other neighbor of $v^+(e_k)$ which is v_i (with $e_k = v_iv_j$) can only increase the size of the independent set. Hence S contains m edge vertices and $s \leqslant n$ original vertices, and there is no larger independent set in H.

We observe that the s original vertices is S form an independent set in G. Indeed, if $v_iv_j = e_k \in E(G)$ and $v_i, v_j \in S$, then neither $v^+(e_k)$ nor $v^-(e_k)$ could be in S.

Now, assume there is an approximation with ratio $\alpha := 1 + \frac{2\beta}{(\Delta+1)^2}$ for Maximum Independent Set on 2-subdivisions running in subexponential time, where $1 + \beta > 1$ is a ratio which is not attainable for Maximum Independent Set on graphs of maximum degree Δ according to Theorem 17. On instance H, this algorithm would output a solution with m' edge vertices and s' original vertices. As we already observed this solution can be easily (in polynomial time) transformed into an at-least-as-good solution with m edge vertices and s'' original vertices forming an independent set in G. Further, we may assume that $s'' \geqslant n/(\Delta+1)$ since for any independent set of G, we can obtain an independent set of H consisting of the same set of original vertices and m edge vertices. Since $m \leqslant n\Delta/2$ and $s'' \geqslant n/(\Delta+1)$, we obtain $m \leqslant s''\Delta(\Delta+1)/2$ and $2m/(\Delta+1)^2 \leqslant s''\Delta/(\Delta+1)$. From $\frac{m+s}{m+s''} \leqslant \alpha$ and $\Delta \geqslant 3$, we have

$$s \leqslant m \cdot \frac{2\beta}{(\Delta+1)^2} + s'' \cdot (1 + \frac{2\beta}{(\Delta+1)^2}) \leqslant s''(\frac{\Delta\beta}{\Delta+1} + 1 + \frac{2\beta}{(\Delta+1)^2}) \leqslant s''(1+\beta)$$

This contradicts the inapproximability of Theorem 17. Indeed, note that the number of vertices of H is only a constant times the number of vertices of G (recall that G has bounded maximum degree, hence $m = O(n)$). ◀

Recalling that independent set is a clique in the complement, we get the following.

▶ **Corollary 19.** *There is a constant $\alpha > 1$ such that for any $\varepsilon > 0$,* Maximum Clique *on the class of all the co-2-subdivisions has no α-approximation algorithm running in subexponential time $2^{n^{1-\varepsilon}}$, unless the ETH fails.*

For exact algorithms the subexponential time that we rule out under the ETH is not only $2^{n^{1-\varepsilon}}$ but actually any $2^{o(n)}$.

Now, to Theorem 15 and Theorem 16, it is sufficient to show that intersection graphs of (filled) ellipses or of (filled) triangles contain all co-2-subdivisions. We start with (filled) triangles since the construction is straightforward.

▶ **Lemma 20.** *The class of intersection graphs of filled triangles contains all co-2-subdivisions.*

Proof. Let G be any graph with n vertices $v_1, \ldots, v_n$ and m edges $e_1, \ldots, e_m$, and H be its co-2-subdivision. We start with $n+2$ points $p_0, p_1, p_2, \ldots, p_n, p_{n+1}$ forming a convex monotone chain. Those points can be chosen as $p_i := (i, p(i))$ where p is the equation of a positive parabola taking its minimum at $(0,0)$. For each $i \in [0, n+1]$, let q_i be the reflection of p_i by the line of equation $y = 0$. Let $x := (n+1, 0)$. For each vertex $v_i \in V(G)$ the filled triangle $\delta_i := p_i q_i x$ encodes v_i. Observe that the points $p_0 = q_0$, p_{n+1}, and q_{n+1} will only be used to define the filled triangles encoding edges.

To encode (the two new vertices of) a subdivided edge $e_k = v_i v_j$, we use two filled triangles Δ_k^+ and Δ_k^-. The triangle Δ_k^+ (resp. Δ_k^-) has an edge which is supported by $\ell(p_{i-1}, p_{i+1})$ (resp. $\ell(q_{j-1}, q_{j+1})$) and is prolonged so that it crosses the boundary of each $\delta_{i'}$ but δ_i (resp. but δ_j). A second edge of Δ_k^+ and Δ_k^- are parallel and make with the horizontal a small angle εk, where $\varepsilon > 0$ is chosen so that εm is smaller than the angle formed by $\ell(p_0, p_1)$ with the horizontal line. Those almost horizontal edges intersect for each pair $\Delta_{k'}^+$ and $\Delta_{k''}^-$ with $k' \neq k''$ intersects close to the same point. Filled triangles Δ_k^+ and Δ_k^- do not intersect. See Figure 2 for the complete picture.

It is easy to check that the intersection graph of $\{\delta_i\}_{i \in [n]} \cup \{\Delta_k^+, \Delta_k^-\}_{k \in [m]}$ is H. The family $\{\delta_i\}_{i \in [n]}$ forms a clique since they all contain for instance the point x. The filled triangle Δ_k^+ (resp. Δ_k^-) intersects every other filled triangles except Δ_k^- (resp. Δ_k^+) and δ_i (resp. δ_j) with $e_k = v_i v_j$.

One may observe that no triangle is fully included in another triangle. So the construction works both as the intersection graph of filled triangles *and* triangles without their interior. The edge of a Δ_k^+ or a Δ_k^- crossing the boundary of all but one δ_i, and the almost horizontal edge can be arbitrary prolonged to the right and to the left respectively. Thus, the triangles can all be made isosceles. ◀

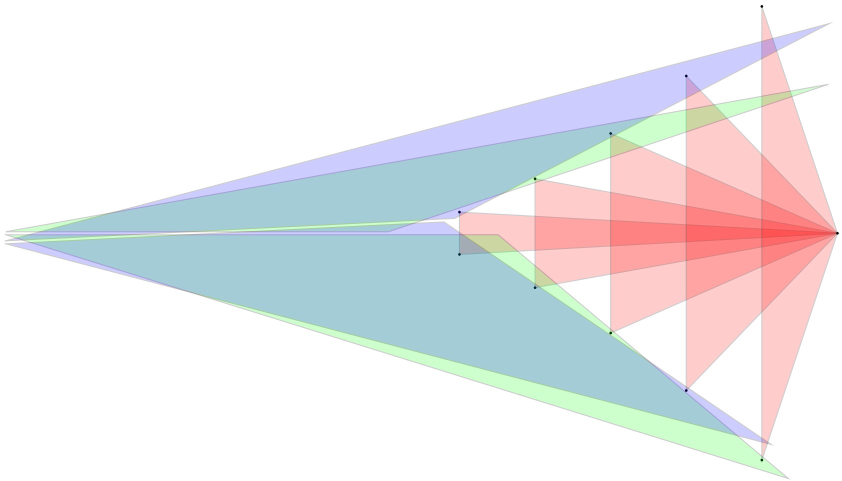

Figure 2 A co-2-subdivision of a graph with 5 vertices (in red) represented with triangles. Only two edges are shown: one between vertices 1 and 4 (green) and one between vertices 2 and 3 (blue).

We use the same ideas for the construction with filled ellipses. The two important sides of a triangle encoding an edge of the initial graph G become two tangents of the ellipse.

▶ **Lemma 21.** *The class of intersection graphs of filled ellipses contains all co-2-subdivisions.*

Proof. Let G be any graph with n vertices $v_1, \ldots, v_n$ and m edges $e_1, \ldots, e_m$, and H be its co-2-subdivision. We start with the convex monotone chain $p_0, p_1, p_2, \ldots, p_{n-1}, p_n, p_{n+1}$, only the gap between p_i and p_{i+1} is chosen very small compared to the positive y-coordinate of p_0. The disks $\mathcal{D}_i$ encoding the vertices $v_i \in G$ must form a clique. We also take p_0 with a large x-coordinate. For $i \in [0, n+1]$, q_i is the symmetric of p_i with respect to the x-axis. For each $i \in [n]$, we define $\mathcal{D}_i$ as the disk whose boundary is the unique circle which goes through p_i and q_i, and whose tangent at p_i has the direction of $\ell(p_{i-1}, p_{i+1})$. It can be observed that, by symmetry, the tangent of $\mathcal{D}_i$ at q_i has the direction of $\ell(q_{i-1}, q_{i+1})$.

Let us call τ_i^+ (resp. τ_i^-) the tangent of $\mathcal{D}_i$ at p_i (resp. at q_i) very slightly translated upward (resp. downward). The tangent τ_i^+ (resp. τ_i^-) intersects every disks $\mathcal{D}_{i'}$ but $\mathcal{D}_i$ (see Figure 3). Let denote by p'_i (resp. q'_i) be the projection of p_i (resp. q_i) onto τ_i^+ (resp. onto τ_i^-) For each $k \in [m]$, let ℓ_k be the line crossing the origin $O = (0, 0)$ and forming with the horizontal an angle εk, where εk is smaller than the angle formed by $\ell(p_0, p_1)$ with the horizontal. Let ℓ_k^+ (resp. ℓ_k^-) be ℓ_k very slightly translated upward (resp. downward). To encode an edge $e_k = v_i v_j$, we have two filled ellipses $\mathcal{E}_k^+$ and $\mathcal{E}_k^-$. The ellipse $\mathcal{E}_k^+$ (resp. $\mathcal{E}_k^-$) is defined as being tangent with τ_i^+ at p'_i (resp. with τ_j^- at q'_j) and tangent at ℓ_k^+ (resp. ℓ_k^-) at the point of x-coordinate 0 (thus very close to O), where $e_k = v_i v_j$. The proof that the intersection graph of $\{\mathcal{D}_i\}_{i \in [n]} \cup \{\mathcal{E}_k^+, \mathcal{E}_k^-\}_{k \in [m]}$ is H is similar to the case of filled triangles.

As no ellipse is fully contained in another ellipse, this construction works for both filled ellipses *and* ellipses without their interior.

We place p_0 at $P := (\sqrt{3}/2, 1/2)$ and make the distance between p_i and p_{i+1} very small compared to 1. All points p_i are very close to P and all points q_i are very close to $Q := (\sqrt{3}/2, -1/2)$. This makes the radius of all disks $\mathcal{D}_i$ arbitrarily close to 1. We choose the convex monotone chain $p_0, \ldots, p_{n+1}$ so that $\ell(p_0, p_1)$ forms a 60-degree angle with the horizontal. As, the chain is strictly convex but very close to a straight-line, $\ell(p_0, p_1) \approx \ell(p_n, p_{n+1}) \approx \ell(p_i, p_{i+1}) \approx \ell(p_i, p_{i+2})$. Thus, all those lines almost cross P and form an angle of roughly 60-degree with the horizontal. The same holds for points q_i. For the choice

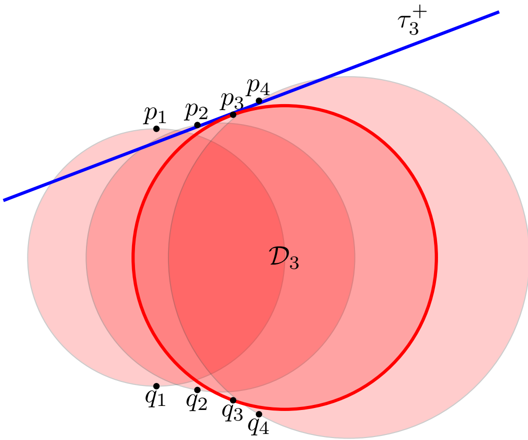

Figure 3 The blue line intersects every red disk but the third one.

of an elliptical disk tangent to the x-axis at O and to a line with a 60-degree slope at P (resp. at Q), we take a disk of radius 1 centered at $(0, 1)$ (resp. at $(0, -1)$); see Figure 4.

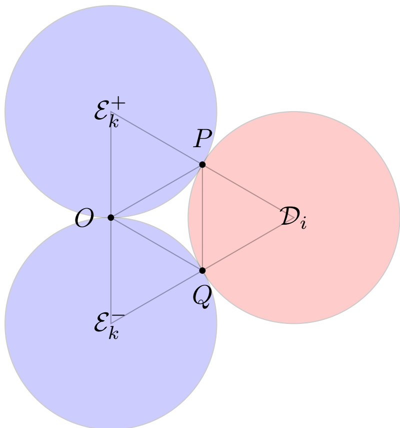

Figure 4 The layout of the disks $\mathcal{D}_i$, and the elliptical disks $\mathcal{E}_k^+$ and $\mathcal{E}_k^-$.

The acute angle formed by ℓ_1 and ℓ_m (incident in O) is made arbitrarily small so that, by continuity of the elliptical disk defined by two tangents at two points, the filled ellipses $\mathcal{E}_k^+$ and $\mathcal{E}_k^-$ have eccentricity arbitrarily close to 0 and major axis arbitrarily close to 1. ◀

In the construction, we made *both* the eccentricity of the (filled) ellipses arbitrarily close to 0 and the ratio between the largest and the smallest major axis arbitrarily close to 1. We know that this construction is very unlikely to work for the extreme case of unit disks, since a polynomial algorithm is known for MAX CLIQUE. Note that even with disks of arbitrary radii, Theorem 6 unconditionally proves that the construction does fail. Indeed the co-2-subdivision of $C_3 + C_3$ is the complement of $C_9 + C_9$, hence not a disk graph.

4.2 Homothets of a convex polygon

Another natural direction of generalizing a result on disk intersection graphs is to consider *pseudodisk intersection graphs*, i.e., intersection graphs of collections of closed subsets of the plane (regions bounded by simple Jordan curves) that are pairwise in a *pseudodisk* relationship (see Kratochvíl [29]). Two regions A and B are in pseudodisk relation if both differences $A \setminus B$ and $B \setminus A$ are arc-connected. It is known that P_{hom} graphs, i.e., intersection graphs of homothetic copies of a fixed polygon P, are pseudodisk intersection graphs [1]. As shown by Brimkov *et al.*, for every convex k-gon P, a P_{hom} graph with n vertices has at most n^k maximal cliques [12]. This clearly implies that Maximum Clique, but also Clique p-Partition for fixed p is polynomially solvable in P_{hom} graphs. Actually, the bound on the maximum number of maximal cliques from [12] holds for a more general class of graphs, called k_{DIR}-CONV, which admit a intersection representation by convex polygons, whose every side is parallel to one of k directions.

Moreover, we observe that Theorem 7 cannot be generalized to P_{hom} graphs or k_{DIR}-CONV graphs. Indeed, consider the complement $\overline{P_n}$ of an n-vertex path P_n. The number of maximal cliques in $\overline{P_n}$, or, equivalently, maximal independent sets in P_n is $\Theta(c^n)$ for $c \approx 1.32$, i.e., exponential in n [22]. Therefore, for every fixed polygon P (or for every fixed k) there is n, such that $\overline{P_n}$ is not a P_{hom} (k_{DIR}-CONV) graph.

5 Perspectives

We presented the first QPTAS and subexponential algorithm for Maximum Clique on disk graphs. Our subexponential algorithm extends to the weighted case and yields a polynomial algorithm if both the degree Δ and the odd girth c of the complement graph are constant. Indeed, our full characterization of disk graphs with co-degree 2, implies a backdoor-to-bipartiteness of size $c\Delta$ in the complement.

We have also paved the way for a potential NP-hardness construction. We showed why the versatile approach of representing complements of even subdivisions of graphs forming a class on which Maximum Independent Set is NP-hard fails if the class is *general graphs*, *planar graphs*, or even any class containing the disjoint union of two odd cycles. This approach was used by Middendorf for some string graphs [32] (with the class of all graphs), Cabello et al. [15] to settle the then long-standing open question of the complexity of Maximum Clique for segments (with the class of planar graphs), in Section 4 of this paper for ellipses and triangles (with the class of all graphs). Determining the complexity of Maximum Independent Set on graphs without two vertex-disjoint odd cycles as an induced subgraph is a valuable first step towards settling the complexity of Maximum Clique on disks.

Another direction is to try and strengthen our QPTAS in one of two ways: either to obtain a PTAS for Maximum Clique on disk graphs, or to obtain a QPTAS (or PTAS) for Maximum Weighted Clique on disk graphs. It is interesting to note that Bock et al. [7] showed a PTAS for Maximum Weighted Independent Set for graphs G with $\mathrm{ocp}(G) = O(\log n / \log \log n)$. However, this bound is too weak to use a win-win approach similar to Theorem 10.

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