Generalised Cantor sets and the dimension of products

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Abstract

In this paper we consider the relationship between the Assouad and box-counting dimension and how both behave under the operation of taking products. We introduce the notion of 'equi-homogeneity' of a set, which requires a uniformity in the cardinality of local covers at all length-scales and at all points, and we show that a large class of homogeneous Moran sets have this property. We prove that the Assouad and box-counting dimensions coincide for sets that have equal upper and lower box-counting dimensions provided that the set 'attains' these dimensions (analogous to 's-sets' when considering the Hausdorff dimension), and the set is equi-homogeneous. Using this fact we show that for any $\alpha \in (0,1)$ and any $\beta, \gamma \in (0,1)$ such that $\beta + \gamma \geq 1$ we can construct two generalised Cantor sets C and D such that $\dim_{\mathrm{B}} C = \alpha\beta$, $\dim_{\mathrm{B}} D = \alpha\gamma$, and $\dim_{\mathrm{A}} C = \dim_{\mathrm{A}} D = \dim_{\mathrm{A}}(C \times D) = \dim_{\mathrm{B}}(C \times D) = \alpha$.

1. *Introduction*

In this paper we study the behaviour of the box-counting and Assouad dimensions (whose definitions we give below) under the action of taking the Cartesian product of sets. Relatively straightforward arguments can be used to show that the Assouad and upper box-counting dimensions satisfy

$$\dim(A \times B) \leq \dim A + \dim B, \tag{1·1}$$

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but constructing examples showing that this inequality is strict is less straightforward. For the box-counting dimension, the first example of sets for which there is strict inequality was constructed in Žubrinić [**21**] who describes two subsets of $\mathbb{R}$ such that the sets and their product have (upper) box-counting dimension one. Later, Robinson & Sharples [**20**] provided examples of sets whose box-counting dimensions take arbitrary values satisfying (1·1): these are Cantor-like sets with carefully controlled ratios, much as those in this paper. A significantly simpler example involving two countable sets followed later from Olson & Robinson [**17**]. For the Assouad dimension, there is an example of strict inequality due to Larman [**11**] (see also Section 9.2 in Robinson [**19**]) of two subsets of $\mathbb{R}$ that accumulate at zero in such a way that the sets and their product all have dimension one.

In this paper we provide a unified treatment of the two dimensions using 'generalised Cantor sets', i.e. Cantor sets in which we allow the portion removed to vary at each stage of the construction in a controlled way. Our argument to calculate the Assouad dimension of generalised Cantor sets and their products relies on the 'equi-homogeneity' of these sets (defined below): roughly this means that there is a uniform bound on the range of the number of balls required in the 'local covers' of the set at each length-scale. In Section 2 we demonstrate that a large class of the much studied homogeneous Moran sets (see Moran [**15**], Feng et al. [**6**], Li et al. [**12**], and Lü et al. [**13**]), which include generalised Cantor sets, are equi-homogeneous. Further, we demonstrate that the product of two equi-homogeneous sets is itself equi-homogeneous.

The equi-homogeneity regularity property is distinct from the measure-theoretic Ahlfors-David regularity property of sets (see David & Semmes [**4**]). An Ahlfors-David regular set necessarily has equal lower box-counting, upper box-counting, Hausdorff and Assouad dimensions (see Li et al. [**12**]), which greatly simplifies the otherwise prohibitively difficult calculation of the Assouad dimension. However, Ahlfors-David regularity is not enjoyed by many important sets, including the generalised Cantor sets considered in Robinson & Sharples [**20**] and Moran sets. In particular, sets with unequal lower and upper box-counting dimensions are not Ahlfors-David regular.

Homogeneous Moran sets satisfy a weaker measure-theoretic regularity property, introduced in Lü et al. [**13**], provided that the contraction ratios that define the Moran set are bounded away from zero. This result is used to calculate the Hausdorff and box-counting dimensions and provide embedding and approximation results for this class of Moran sets. In contrast, the equi-homogeneity property introduced in this paper requires no restriction on the contraction ratios of Moran sets, making it a natural notion of regularity for homogeneous Moran sets.

We discuss equi-homogeneity in a more general setting in Henderson et al. [**9**] where we prove that the attractors of a large class of iterated function systems are equi-homogeneous. However, the arguments presented here will serve as prototypes for the more general results in Henderson et al. [**9**].

1·1. *Counting covers*

We begin by defining some notions of dimension for subsets of a metric space (X, d_X). For a set $F \subset X$ and a length $\delta > 0$ we denote by $\mathcal{D}(F, \delta)$ the minimum number of sets of diameter δ that cover F, which is to say that F is contained in their union, where the diameter of a set A is given by $\mathrm{diam}\,(A) = \sup\{\mathrm{d}_X\,(x, y) : \ x, y \in A\}$. If $\mathcal{D}(F, \delta)$ is finite for all $\delta > 0$ we say that the set F is *totally bounded*.

There are many similar geometric quantities, some of which we will make use of in what follows:

- $\mathcal{N}(F,\delta)$, the minimum number of balls of radius δ ('δ-balls') with centres in F required to cover F;
- $\mathcal{P}(F,\delta)$, the maximum number of disjoint δ-balls with centres in F.

It is a short exercise to establish that these geometric quantities satisfy

$$\mathcal{D}(F,4\delta) \leq \mathcal{N}(F,2\delta) \leq \mathcal{P}(F,\delta) \leq \mathcal{D}(F,\delta) \tag{1·2}$$

(see, for example, 'Equivalent definitions' 2.1 in Falconer [**5**] or Lemma 2.1 in Robinson & Sharples [**20**][1]).

We adopt the cover by centred δ-balls as our primary measure since it is convenient for sets of the form $B_\delta(x) \cap F$ with $x \in F$, which feature in the definition of the Assouad dimension.

We recall that for each $\delta > 0$ the function $\mathcal{N}(\cdot,\delta)$ is

- monotonic, that is $A \subset B \Rightarrow \mathcal{N}(A,\delta) \leq \mathcal{N}(B,\delta)$, and
- subadditive, that is $\mathcal{N}(A \cup B,\delta) \leq \mathcal{N}(A,\delta) + \mathcal{N}(B,\delta)$,

and that for each set $F \subset X$ the function $\mathcal{N}(F,\cdot)$ is non-increasing.

1·2. *Box-Counting Dimension*

First, we recall the definition of the familiar box-counting dimensions.

Definition 1·1. *For a totally bounded set $F \subset X$ we define the lower and upper box-counting dimensions of F as the quantities*

$$\dim_{\mathrm{LB}} F := \liminf_{\delta\to 0+} \frac{\log \mathcal{N}(F,\delta)}{-\log\delta},$$

$$\textit{and} \qquad \dim_{\mathrm{B}} F := \limsup_{\delta\to 0+} \frac{\log \mathcal{N}(F,\delta)}{-\log\delta}$$

respectively.

In light of the inequalities (1·2), replacing $\mathcal{N}(F,\delta)$ with any of the geometric quantities mentioned above gives an equivalent definition. The box-counting dimensions essentially capture the exponent $s \in \mathbb{R}^+$ for which the minimum number of centred δ-balls required to cover F scales like $\mathcal{N}(F,\delta) \sim \delta^{-s}$. More precisely, it follows from Definition 1·1 that for all $\delta_0 > 0$ and any $\varepsilon > 0$ there exists a constant $C \geq 1$ such that

$$C^{-1}\delta^{-\dim_{\mathrm{LB}} F+\varepsilon} \leq \mathcal{N}(F,\delta) \leq C\delta^{-\dim_{\mathrm{B}} F-\varepsilon} \qquad \text{for all } 0 < \delta \leq \delta_0. \tag{1·3}$$

In some cases the bounds (1·3) will also hold at the limit $\varepsilon \to 0$, that is for each $\delta_0 > 0$ there exists a constant $C \geq 1$ such that

$$\frac{1}{C}\delta^{-\dim_{\mathrm{LB}} F} \leq \mathcal{N}(F,\delta) \leq C\delta^{-\dim_{\mathrm{B}} F} \qquad \text{for all } 0 < \delta \leq \delta_0, \tag{1·4}$$

giving precise control of the growth of $\mathcal{N}(F,\delta)$. We distinguish this class of sets in the following definition:

[1] Both Falconer [**5**] and Robinson & Sharples [**20**] instead consider $\mathcal{N}(F,\delta)$ to be the minimum number of *arbitrary* δ-balls required to cover F. However (1·2) follows immediately from the argument given by these authors.

Definition 1·2. *We say that a bounded set* $F \subset X$ attains *its lower box-counting dimension if for all* $\delta_0 > 0$ *there exists a positive constant* $C \le 1$ *such that*

$$\mathcal{N}(F,\delta) \ge C\delta^{-\dim_{\mathrm{LB}} F} \qquad \textit{for all} \quad 0 < \delta \le \delta_0.$$

Similarly, we say that F attains *its upper box-counting dimension if for all* $\delta_0 > 0$ *there exists a constant* $C \ge 1$ *such that*

$$\mathcal{N}(F,\delta) \le C\delta^{-\dim_{\mathrm{B}} F} \qquad \textit{for all} \quad 0 < \delta \le \delta_0.$$

We remark that a similar distinction is made with regard to the Hausdorff dimension of sets: recall that the Hausdorff measures are a one-parameter family of measures, denoted $\mathcal{H}^s$ with parameter $s \in \mathbb{R}^+$, and that for each set $F \subset \mathbb{R}^n$ there exists a value $\dim_{\mathrm{H}} F \in \mathbb{R}^+$, called the Hausdorff dimension of F, such that

$$\mathcal{H}^s(F) = \begin{cases} \infty & s < \dim_{\mathrm{H}} F \\ 0 & s > \dim_{\mathrm{H}} F. \end{cases}$$

For a set F to have Hausdorff dimension d it is sufficient, but not necessary, for the Hausdorff measure with parameter d to satisfy $0 < \mathcal{H}^d(F) < \infty$. Sets with this property are sometimes called d-sets (see, for example, Falconer [**5**] pp.48) and are distinguished as they have many convenient properties. For example, the Hausdorff dimension product formula $\dim_{\mathrm{H}}(F \times G) \ge \dim_{\mathrm{H}} F + \dim_{\mathrm{H}} G$ was first proved for sets F and G in this restricted class (see Besicovitch & Moran [**2**]) before being extended to hold for all sets (see Howroyd [**10**]).

1·3. *Homogeneity and the Assouad Dimension*

The Assouad dimension is a less familiar notion of dimension, in which we are concerned with 'local' coverings of a set F: for more details see Assouad [**1**], Bouligand [**3**], Fraser [**7**], Luukkainen [**14**], Olson [**16**], or Robinson [**19**].

Definition 1·3. *A set* $F \subset X$ *is s-homogeneous if for all* $\delta_0 > 0$ *there exists a constant* $C > 0$ *such that*

$$\sup_{x\in F} \mathcal{N}(B_\delta(x) \cap F, \rho) \le C(\delta/\rho)^s \qquad \textit{for all } \delta, \rho \textit{ with } 0 < \rho < \delta \le \delta_0.$$

Note that we do not require a set to be totally bounded in order for it to be s-homogeneous.

Definition 1·4. *The Assouad dimension of a set* $F \subset \mathbb{R}^n$ *is defined by*

$$\dim_{\mathrm{A}} F := \inf\left\{s \in \mathbb{R}^+ : \ F \textit{ is s-homogeneous}\right\}$$

It is known that for a totally bounded set $F \subset X$ the three notions of dimension that we have now introduced satisfy

$$\dim_{\mathrm{LB}} F \le \dim_{\mathrm{B}} F \le \dim_{\mathrm{A}} F \qquad (1{\cdot}5)$$

(see, for example, Lemma 9.6 in Robinson [**19**] or Lemma 1.9 of Henderson et al. [**9**]). An interesting example is given by the compact countable set $F_\alpha := \{n^{-\alpha}\}_{n\in\mathbb{N}} \cup \{0\} \subset \mathbb{R}$ with $\alpha > 0$ for which

$$\dim_{\mathrm{LB}} F_\alpha = \dim_{\mathrm{B}} F_\alpha = (1+\alpha)^{-1}$$
$$\text{but} \qquad \dim_{\mathrm{A}} F_\alpha = 1.$$

(see Olson [**16**] and Example 13.4 in Robinson [**18**]).

1·4. *Product Sets*

Let (X, d_X) and (Y, d_Y) be metric spaces and endow the product space $X \times Y$ with a metric $\mathrm{d}_{X\times Y}$ that satisfies

$$m_1 \max\left(\mathrm{d}_X, \mathrm{d}_Y\right) \leq \mathrm{d}_{X\times Y} \leq m_2 \max\left(\mathrm{d}_X, \mathrm{d}_Y\right) \tag{1·6}$$

for some $m_1, m_2 > 0$ with $m_1 \leq m_2$. Clearly the familiar product metric

$$\mathrm{d}_{X\times Y,\infty} := \max\left(\mathrm{d}_X, \mathrm{d}_Y\right)$$

satisfies (1·6), as do the metrics

$$\mathrm{d}_{X\times Y,p} := \left(\mathrm{d}_X^p + \mathrm{d}_Y^p\right)^{\frac{1}{p}} \qquad \text{for } p \in [1, \infty)$$

with $m_1 = 1$ and $m_2 = 2^{\frac{1}{p}}$.

It is well known that if $F \subset X$ and $G \subset Y$ are two totally bounded sets then the box-counting and Assouad dimensions of their product $F \times G \subset X \times Y$ satisfy

$$\dim_{\mathrm{LB}}\left(F \times G\right) \geq \dim_{\mathrm{LB}} F + \dim_{\mathrm{LB}} G \tag{1·7}$$

$$\dim_{\mathrm{B}}\left(F \times G\right) \leq \dim_{\mathrm{B}} F + \dim_{\mathrm{B}} G \tag{1·8}$$

$$\text{and} \qquad \dim_{\mathrm{A}}\left(F \times G\right) \leq \dim_{\mathrm{A}} F + \dim_{\mathrm{A}} G. \tag{1·9}$$

provided that the product metric $\mathrm{d}_{X\times Y}$ satisfies (1·6) (see, for example, 'Product formula 7.5' of Falconer [**5**], and §9.2 of Robinson [**19**]).

The box-counting dimension product formulae were improved in Robinson & Sharples [**20**] who demonstrate that product sets satisfy the chain of inequalities

$$\begin{aligned}
\dim_{\mathrm{LB}} F + \dim_{\mathrm{LB}} G &\leq \dim_{\mathrm{LB}}\left(F \times G\right) \\
&\leq \min\left\{\dim_{\mathrm{LB}} F + \dim_{\mathrm{B}} G, \dim_{\mathrm{B}} F + \dim_{\mathrm{LB}} G\right\} \\
&\leq \max\left\{\dim_{\mathrm{LB}} F + \dim_{\mathrm{B}} G, \dim_{\mathrm{B}} F + \dim_{\mathrm{LB}} G\right\} \\
&\leq \dim_{\mathrm{B}}\left(F \times G\right) \leq \dim_{\mathrm{B}} F + \dim_{\mathrm{B}} G,
\end{aligned} \tag{1·10}$$

and that paper provides a method for constructing sets so that their box-counting dimensions can take arbitrary values satisfying this chain of inequalities.

We remark that if $\dim_{\mathrm{LB}} F = \dim_{\mathrm{B}} F$ then it follows from (1·10) that there is equality in (1·7) and (1·8), so the good behaviour of just one set guarantees equality in the box-counting product formulas.

The box-counting dimension product formulae in (1·10) are all consequences of the geometric inequalities

$$\begin{aligned}
&\mathcal{N}\big(F \times G, m_2\delta\big) \leq \mathcal{N}(F,\delta)\mathcal{N}(G,\delta) \\
\text{and} \qquad &\mathcal{P}\left(F \times G, m_1\delta\right) \geq \mathcal{P}\left(F,\delta\right)\mathcal{P}\left(G,\delta\right),
\end{aligned}$$

which in turn follow from the inclusions

$$B_{\delta/m_2}\left(x\right) \times B_{\delta/m_2}\left(y\right) \subset B_\delta\left(\left(x,y\right)\right) \subset B_{\delta/m_1}\left(x\right) \times B_{\delta/m_1}\left(y\right), \tag{1·11}$$

as the product of centred δ-ball covers of F and G gives rise to a centred $m_2\delta$-ball cover of $F \times G$, and the product of disjoint δ-balls with centres in F and G gives rise to a set of disjoint $m_1\delta$-balls with centres in $F \times G$ (see, for example, Falconer [**5**] or

Robinson & Sharples [**20**] for further details). Combining these product inequalities with the relationships in (1·2) we obtain the expression

$$\mathcal{N}(F,4\delta/m_1)\mathcal{N}(G,4\delta/m_1) \leq \mathcal{N}(F\times G,\delta) \leq \mathcal{N}(F,\delta/m_2)\mathcal{N}(G,\delta/m_2) \tag{1·12}$$

which will be useful in the remainder.

To establish the Assouad dimension product inequality we prove the following geometric relationship. One can find a very similar argument for both bounds in Olson [**16**] (Theorem 3.2) and for the upper bound in Robinson [**19**] (Lemma 9.7).

Lemma 1·5. *If $F\subset X$ and $G\subset Y$ then for all $\mathbf{x}=(x,y)\in F\times G$ and all $\delta,\rho>0$*

$$\mathcal{N}\left(B_\delta(\mathbf{x})\cap(F\times G),\rho\right) \leq \mathcal{N}\left(B_{\delta/m_1}(x)\cap F,\rho/m_2\right)\mathcal{N}\left(B_{\delta/m_1}(y)\cap G,\rho/m_2\right)$$

and

$$\mathcal{N}\left(B_\delta(\mathbf{x})\cap(F\times G),\rho\right) \geq \mathcal{N}\left(B_{\delta/m_2}(x)\cap F,4\rho/m_1\right)\mathcal{N}\left(B_{\delta/m_2}(y)\cap G,4\rho/m_1\right).$$

Proof. From (1·11) it follows that

$$\begin{aligned} & B_\delta(\mathbf{x})\cap(F\times G)\subset\left(B_{\delta/m_1}(x)\cap F\right)\times\left(B_{\delta/m_1}(y)\cap G\right)\\ \text{and}\qquad & B_\delta(\mathbf{x})\cap(F\times G)\supset\left(B_{\delta/m_2}(x)\cap F\right)\times\left(B_{\delta/m_2}(y)\cap G\right).\end{aligned}$$

Consequently, as the function $\mathcal{N}(\cdot,\rho)$ is monotonic, it follows from (1·12) that

$$\begin{aligned}\mathcal{N}\left(B_\delta(\mathbf{x})\cap(F\times G),\rho\right) &\leq \mathcal{N}\left(\left(B_{\delta/m_1}(x)\cap F\right)\times\left(B_{\delta/m_1}(y)\cap G\right),\rho\right)\\ &\leq \mathcal{N}\left(B_{\delta/m_1}(x)\cap F,\rho/m_2\right)\mathcal{N}\left(B_{\delta/m_1}(y)\cap G,\rho/m_2\right),\end{aligned}$$

and

$$\begin{aligned}\mathcal{N}\left(B_\delta(\mathbf{x})\cap(F\times G),\rho\right) &\geq \mathcal{N}\left(\left(B_{\delta/m_2}(x)\cap F\right)\times\left(B_{\delta/m_2}(y)\cap G\right),\rho\right)\\ &\geq \mathcal{N}\left(B_{\delta/m_2}(x)\cap F,4\rho/m_1\right)\mathcal{N}\left(B_{\delta/m_2}(y)\cap G,4\rho/m_1\right)\end{aligned}$$

as required. □

It is now simple to prove the following Assouad dimension formula for products. We remark that in Olson [**16**], Theorem 3.2, it was mistakenly asserted that equality holds in this product formula. However, the argument there (which we reproduce here) shows that equality *does* hold for products of the form $F\times F$.

Lemma 1·6. *If $F\subset X$ and $G\subset Y$ then*

$$\dim_A(F\times G)\leq\dim_A F+\dim_A G \tag{1·13}$$

and

$$\dim_A(F\times F)=2\dim_A F. \tag{1·14}$$

Proof. Fix $\delta_0>0$. If F is an s-homogeneous set and G is a t-homogeneous set then from Lemma 1·5 it follows that for all δ,ρ with $0<\rho<\delta\leq\delta_0$

$$\mathcal{N}(B_\delta(\mathbf{x})\cap(F\times G),\rho) \leq \mathcal{N}\left(B_{\delta/m_1}(x)\cap F,\rho/m_2\right)\mathcal{N}\left(B_{\delta/m_1}(y)\cap G,\rho/m_2\right).$$

Therefore, since the sets F and G are homogeneous and $0 < \rho/m_2 < \delta/m_1 \leq \delta_0/m_1$, there exist constants $C_F, C_G > 0$ so that

$$\begin{aligned} &\leq C_F C_G \left(\frac{\delta/m_1}{\rho/m_2}\right)^s \left(\frac{\delta/m_1}{\rho/m_2}\right)^t \\ &\leq C_F C_G \left(m_2/m_1\right)^{s+t} \left(\delta/\rho\right)^{s+t}. \end{aligned}$$

As $\delta_0 > 0$ was arbitrary we conclude that the set $F \times G$ is $(s+t)$-homogeneous, from which we obtain (1·13).

Now suppose that $F = G$. Given $\epsilon > 0$, find $x \in F$ such that

$$\mathcal{N}(B_\delta(x) \cap F, \rho) \geq C(\delta/\rho)^{s-\epsilon}$$

for some $0 < \rho < \delta$. Then for $\mathbf{x} = (x,x) \in F \times F$ we have

$$\begin{aligned} \mathcal{N}\left(B_{m_2\delta}(\mathbf{x}) \cap (F \times F), m_1\rho/4\right) &\geq \mathcal{N}\left(B_\delta(x) \cap F, \rho\right) \mathcal{N}\left(B_\delta(x) \cap G, \rho\right) \\ &\geq C^2 (\delta/\rho)^{2(s-\epsilon)}; \end{aligned}$$

it follows that $\dim_A(F \times F) \geq 2(s-\epsilon)$ for every $\epsilon > 0$, which yields (1·14). $\square$

2. *Equi-homogeneous sets*

From Definition 1·3 we see that homogeneity encodes the *maximum* cardinality of a local optimal cover at a particular length-scale. However, the *minimal* cardinality of a local optimal cover is not captured by homogeneity, and indeed this minimum cardinality can scale very differently, as the set described in Section 1·3 illustrates.

EXAMPLE 2·1. *For each $\alpha > 0$ the set $F_\alpha := \{n^{-\alpha}\}_{n\in\mathbb{N}} \cup \{0\}$ has Assouad dimension equal to* 1*, so for all $\varepsilon > 0$*

$$\sup_{x \in F_\alpha} \mathcal{N}(B_\delta(x) \cap F_\alpha, \rho)(\delta/\rho)^{-(1-\varepsilon)}$$

is unbounded on δ, ρ with $0 < \rho < \delta$.

On the other hand $1 \in F_\alpha$ is an isolated point so

$$\inf_{x \in F_\alpha} \mathcal{N}(B_\delta(x) \cap F_\alpha, \rho) = 1 \qquad \text{for all } \delta, \rho \text{ with } 0 < \rho < \delta < 1 - 2^{-\alpha} \tag{2·1}$$

as $B_\delta(1) \cap F_\alpha = \{1\}$ for such δ and this isolated point can be covered by a single ball of any radius.

It is worth remarking that the lower Assouad dimension of F_α, denoted $\dim_{\mathrm{LA}} F_\alpha$, is zero: the quantity $\dim_{\mathrm{LA}} F$, also called the minimal dimension, was defined in Larman [**11**] as the supremum over all s for which there exists constants c and δ_0 such that

$$\inf_{x \in F} \mathcal{N}\left(B_\delta(x) \cap F, \rho\right) \geq c\,(\delta/\rho)^s \qquad \text{for all } \delta, \rho \text{ with } 0 < \rho < \delta \leq \delta_0. \tag{2·2}$$

For the set F_α it is immediate from (2·1) that (2·2) holds only for $s = 0$, hence $\dim_{\mathrm{LA}} F_\alpha = 0$. This argument is easily adapted to demonstrate that $\dim_{\mathrm{LA}} F = 0$ for any set F containing an isolated point.

The minimal and Assouad dimensions form a pair that respectively provide lower and upper bounds on the scaling of the quantity $\mathcal{N}\left(B_\delta(x) \cap F, \rho\right)$, which is analogous to the lower and upper box-counting dimensions providing bounds on the scaling on $\mathcal{N}(F, \delta)$ in (1·3).

The equi-homogeneity property (defined below) holds if the quantity of interest for the minimal dimension, $\inf_{x\in F}\mathcal{N}(B_\delta(x)\cap F,\rho)$, scales identically (up to a constant) to the quantity of interest for the Assouad dimension $\sup_{x\in F}\mathcal{N}(B_\delta(x)\cap F,\rho)$. In fact, it follows that both of these quantities scale identically to $\mathcal{N}(B_\delta(x)\cap F,\rho)$ for any $x\in F$, which we can interpret as equi-homogeneous sets having identical fractal detail at every point.

However it does not follow that the minimal and Assouad dimensions are equal for equi-homogeneous sets: the upper bound on the scaling of $\mathcal{N}(B_\delta(x)\cap F,\rho)$ may differ from the lower bound if the cardinality of local covers oscillates rapidly as we change the length-scales. See Henderson et al. [**9**] for a detailed example.

For a totally bounded set the maximal and minimal cardinality of local optimal covers can be estimated by more elementary quantities.

Lemma 2·2. *For a totally bounded set $F\subset X$ and δ,ρ satisfying* $0<\rho<\delta$

$$\inf_{x\in F}\mathcal{N}(B_\delta(x)\cap F,\rho)\le\frac{\mathcal{N}(F,\rho)}{\mathcal{N}(F,4\delta)} \tag{2·3}$$

$$\text{and}\qquad \sup_{x\in F}\mathcal{N}(B_\delta(x)\cap F,\rho)\ge\frac{\mathcal{N}(F,\rho)}{\mathcal{N}(F,\delta)}. \tag{2·4}$$

Proof. Let $x_1,\ldots,x_{\mathcal{N}(F,\delta)}\in F$ be the centres of δ-balls that form a cover of F. Clearly,

$$\mathcal{N}(F,\rho)\le\sum_{j=1}^{\mathcal{N}(F,\delta)}\mathcal{N}(B_\delta(x_j)\cap F,\rho)\le\mathcal{N}(F,\delta)\sup_{x\in F}\mathcal{N}(B_\delta(x)\cap F,\rho),$$

which is (2·4).

Next, let δ,ρ satisfy $0<\rho<\delta$ and let $x_1,\ldots,x_{\mathcal{P}(F,4\delta)}\in F$ be the centres of disjoint 4δ-balls. Observe that an arbitrary ρ-ball $B_\rho(z)$ intersects at most one of the balls $B_\delta(x_i)$: indeed, if there exist $x,y\in B_\rho(z)$ with $x\in B_\delta(x_i)$ and $y\in B_\delta(x_j)$ with $i\ne j$ then

$$d_X(x_i,x_j)\le d_X(x_i,x)+d_X(x,z)+d_X(z,y)+d_X(y,x_j)\le 2\delta+2\rho\le 4\delta$$

and so $x_i\in B_{4\delta}(x_j)$, which is a contradiction. Consequently, as F contains the union $\bigcup_{j=1}^{\mathcal{P}(F,4\delta)}B_\delta(x_j)\cap F$, it follows that

$$\begin{aligned}\mathcal{N}(F,\rho)&\ge\sum_{j=1}^{\mathcal{P}(F,4\delta)}\mathcal{N}(B_\delta(x_j)\cap F,\rho)\\&\ge\mathcal{P}(F,4\delta)\inf_{x\in F}\mathcal{N}(B_\delta(x)\cap F,\rho),\\&\ge\mathcal{N}(F,4\delta)\inf_{x\in F}\mathcal{N}(B_\delta(x)\cap F,\rho)\end{aligned}$$

from (1·2), which is precisely (2·3). □

In contrast there is, in general, no similar elementary upper bound on the quantity $\sup_{x\in F}\mathcal{N}(B_\delta(x)\cap F,\rho)$, the existence of which would be useful in determining the Assouad dimension of F. For this reason we introduce the notion of equi-homogeneity. A set is equi-homogeneous if the maximal cardinality of local covers at one length-scale can be bounded by the minimal cardinality of local covers at another length-scale in a uniform way.

Definition 2·3. *We say that a set $F\subset X$ is* equi-homogeneous *if for all $\delta_0>0$ there* *exist constants* $M \geq 1$, *and* $\lambda_1, \lambda_2 > 0$ *such that*

$$\sup_{x\in F} \mathcal{N}\left(B_\delta(x) \cap F, \rho\right) \leq M \inf_{x\in F} \mathcal{N}\left(B_{\lambda_1\delta}(x) \cap F, \lambda_2\rho\right) \tag{2·5}$$

for all δ, ρ *with* $0 < \rho < \delta \leq \delta_0$.

As with the definition of the box-counting dimensions, it follows from the geometric inequalities (1·2) that replacing $\mathcal{N}$ with the geometric quantities $\mathcal{P}$ or $\mathcal{D}$ gives an equivalent definition of equi-homogeneity. Further, note that as $\mathcal{N}(B_\delta(x) \cap F, \rho)$ increases with δ and decreases with ρ we can assume that $\lambda_2 \leq \lambda_1$ in (2·5). If a totally bounded set F is equi-homogeneous then, in addition to the lower bound (2·4), we can find an upper bound for the maximal size of the local coverings.

Corollary 2·4. *If* $F \subset X$ *is totally bounded and equi-homogeneous then for all* $\delta_0 > 0$ *there exist constants* $M \geq 1$ *and* $\lambda_1, \lambda_2 > 0$ *with* $\lambda_2 \leq \lambda_1$ *such that*

$$\frac{\mathcal{N}(F,\rho)}{\mathcal{N}(F,\delta)} \leq \sup_{x\in F} \mathcal{N}\left(B_\delta(x) \cap F, \rho\right) \leq M \inf_{x\in F} \mathcal{N}\left(B_{\lambda_1\delta}(x) \cap F, \lambda_2\rho\right) \leq M \frac{\mathcal{N}(F,\lambda_2\rho)}{\mathcal{N}(F,4\lambda_1\delta)}$$

for all $0 < \rho < \delta \leq \delta_0$.

Proof. The corollary immediately follows from Definition 2·3 and Lemma 2·2 as $0 < \lambda_2\rho < \lambda_1\delta$ if $0 < \lambda_2 \leq \lambda_1$ and $0 < \rho < \delta$. □

In fact, with this bound we can precisely find the Assouad dimension of equi-homogeneous sets provided that their box-counting dimensions are suitably 'well behaved'.

Theorem 2·5. *If a set* $F \subset X$ *is equi-homogeneous, attains both its upper and lower box-counting dimensions, and* $\dim_{\mathrm{LB}} F = \dim_{\mathrm{B}} F$, *then*

$$\dim_{\mathrm{A}} F = \dim_{\mathrm{B}} F = \dim_{\mathrm{LB}} F.$$

Proof. Assume that F satisfies the above hypotheses and fix $\delta_0 > 0$. As F attains both its upper and lower box-counting dimensions and these dimensions are equal it is clear from Definition 1·2 that there exists a constant $C \geq 1$ such that

$$\frac{1}{C}\delta^{-\dim_{\mathrm{B}} F} \leq \mathcal{N}(F,\delta) \leq C\delta^{-\dim_{\mathrm{B}} F} \qquad \text{for all } 0 < \delta \leq \delta_0. \tag{2·6}$$

Next, as F is equi-homogeneous it follows from Corollary 2·4 that

$$\sup_{x\in F} \mathcal{N}\left(B_\delta(x) \cap F, \rho\right) \leq M \frac{\mathcal{N}(F,\lambda_2\rho)}{\mathcal{N}(F,4\lambda_1\delta)} \qquad \text{for all } 0 < \rho < \delta \leq \delta_0$$

for some constants $M \geq 1$ and $\lambda_1, \lambda_2 > 0$, which from (2·6)

$$\begin{aligned} &\leq MC^2 \frac{(\lambda_2\rho)^{-\dim_{\mathrm{B}} F}}{(4\lambda_1\delta)^{-\dim_{\mathrm{B}} F}} \\ &= MC^2 (4\lambda_1/\lambda_2)^{\dim_{\mathrm{B}} F} (\delta/\rho)^{\dim_{\mathrm{B}} F} \end{aligned}$$

for all δ, ρ with $0 < \rho < \delta \leq \delta_0$, so the set F is $(\dim_{\mathrm{B}} F)$-homogeneous. Consequently, $\dim_{\mathrm{A}} F \leq \dim_{\mathrm{B}} F$, but from (1·5) the Assouad dimension dominates the upper box-counting dimension so we obtain the equality $\dim_{\mathrm{A}} F = \dim_{\mathrm{B}} F$. □

2·1. *Product sets*

The generalised Cantor sets introduced in the next section are the prototypical examples of equi-homogeneous sets, and it is precisely these sets that we use to construct examples of strict inequality in the Assouad dimension product formula. In this construction we will determine the Assouad dimension of the product of generalised Cantor sets by applying the above theorem, which first requires us to show that the product set is equi-homogeneous. However, this immediately follows from the fact that equi-homogeneity is preserved upon taking products, which we now prove.

Lemma 2·6. *If $F \subset X$ and $G \subset Y$ are equi-homogeneous and the product space $X \times Y$ is endowed with a metric satisfying* (1·6)*, then the product $F \times G \subset X \times Y$ is equi-homogeneous.*

Proof. Fix $\delta_0 > 0$. As F and G are equi-homogeneous, there exist constants $M_F, M_G \geq 1$ and $f_1, f_2, g_1, g_2 > 0$ such that for all $0 < \rho < \delta \leq \delta_0/m_1$

$$\begin{aligned}\sup_{x\in F} \mathcal{N}\left(B_\delta\left(x\right)\cap F, \rho\right) &\leq M_F \inf_{x\in F} \mathcal{N}\left(B_{f_1\delta}\left(x\right)\cap F, f_2\rho\right) \\ &\leq M_F \inf_{x\in F} \mathcal{N}\left(B_{\lambda_1\delta}\left(x\right)\cap F, \lambda_2\rho\right)\end{aligned} \tag{2·7}$$

and

$$\begin{aligned}\sup_{y\in G} \mathcal{N}\left(B_\delta\left(y\right)\cap G, \rho\right) &\leq M_G \inf_{y\in G} \mathcal{N}\left(B_{g_1\delta}\left(y\right)\cap G, g_2\rho\right) \\ &\leq M_G \inf_{y\in G} \mathcal{N}\left(B_{\lambda_1\delta}\left(y\right)\cap G, \lambda_2\rho\right),\end{aligned} \tag{2·8}$$

where $\lambda_1 = \max\left(f_1, g_1\right)$ and $\lambda_2 = \min\left(f_2, g_2\right)$, and the second inequalities follow from the monotonicity of $\mathcal{N}\left(\cdot, \rho\right)$ and the fact that $\mathcal{N}\left(A, \cdot\right)$ is non-increasing.

Now, from Lemma 1·5 for all $0 < \rho < \delta \leq \delta_0$

$$\begin{aligned}N_{F\times G}\left(\delta, \rho\right) &:= \sup_{\mathbf{x}\in F\times G} \mathcal{N}\left(B_\delta(\mathbf{x})\cap\left(F\times G\right), \rho\right) \\ &\leq \left[\sup_{x\in F} \mathcal{N}\left(B_{\delta/m_1}(x)\cap F, \rho/m_2\right)\right]\left[\sup_{y\in G} \mathcal{N}\left(B_{\delta/m_1}(y)\cap G, \rho/m_2\right)\right]\end{aligned}$$

as taking suprema is submultiplicative. Since $0 < \rho/m_2 < \delta/m_1 \leq \delta_0/m_1$ it follows from (2·7) and (2·8) that $N_{F\times G}\left(\delta, \rho\right)$ is bounded above by

$$\begin{aligned}&\left[M_F \inf_{x\in F} \mathcal{N}\left(B_{\lambda_1\delta/m_1}(x)\cap F, \lambda_2\rho/m_2\right)\right]\left[M_G \inf_{y\in G} \mathcal{N}\left(B_{\lambda_1\delta/m_1}(y)\cap G, \lambda_2\rho/m_2\right)\right] \\ &\leq M_F M_G \inf_{(x,y)\in F\times G} \mathcal{N}\left(B_{\lambda_1\delta/m_1}(x)\cap F, \lambda_2\rho/m_2\right)\mathcal{N}\left(B_{\lambda_1\delta/m_1}(y)\cap G, \lambda_2\rho/m_2\right),\end{aligned}$$

as taking infima is supermultiplicative. Again applying Lemma 1·5 we obtain the upper bound

$$N_{F\times G}\left(\delta, \rho\right) \leq M_F M_G \inf_{\mathbf{x}\in F\times G} \mathcal{N}\left(B_{\frac{\lambda_1 m_2}{m_1}\delta}(\mathbf{x})\cap\left(F\times G\right), \tfrac{\lambda_2 m_1}{4m_2}\rho\right)$$

for all $0 < \rho < \delta \leq \delta_0$ and as $\delta_0 > 0$ was arbitrary we conclude that $F \times G$ is equi-homogeneous. $\square$

2·2. *Homogeneous Moran sets*

We now demonstrate that homogeneous Moran sets are equi-homogeneous. Moran sets were introduced in Moran [**15**] and remain a frequently studied class of fractal sets. We recall the definition of homogeneous Moran sets (see, for example, Feng et al. [**6**]).

Let $J \subset \mathbb{R}^d$ be a compact set with non-empty interior. Let $\{n_k\}_{k\in\mathbb{N}}$ be a sequence of integers with $n_k \geq 2$ and $\{c_k\}_{k\in\mathbb{N}}$ be a sequence of contraction ratios $c_k \in (0,1)$ with $n_k(c_k)^d \leq 1$ for each $k \in \mathbb{N}$. For each $k \in \mathbb{N}$ define $D_k = \{u_1u_2\cdots u_k | 1 \leq u_j \leq n_j, j \leq k\}$ to be the set of words of length k, let $D_0 = \{\emptyset\}$ be the set consisting of the empty word $\emptyset$, and let $D = \cup_{k=0}^{\infty} D_k$ be the set of all finite words. We say that a collection of sets $\{J_{\mathbf{u}} | \mathbf{u} \in D\}$ with $J_\emptyset = J$ satisfies the *Moran structure conditions* if

(i) for each $\mathbf{u} \in D$ there exists a similarity $S_{\mathbf{u}} \colon \mathbb{R}^d \to \mathbb{R}^d$ such that $J_{\mathbf{u}} = S_{\mathbf{u}}(J)$,

(ii) for all $k \in \mathbb{N}$ and all $\mathbf{u} \in D_{k-1}$ the k^{th}-level sets $J_{\mathbf{u}1}, \ldots, J_{\mathbf{u}n_k}$

  (a) are the k^{th}-level subsets of $J_{\mathbf{u}}$, that is $J_{\mathbf{u}i} \subset J_{\mathbf{u}}$ for $i = 1, \ldots, n_k$ and, for all $\mathbf{v} \in D_k$, $J_{\mathbf{v}} \subset J_{\mathbf{u}} \Rightarrow \mathbf{v} = \mathbf{u}i$ for some $i = 1, \ldots, n_k$,

  (b) have pairwise disjoint interiors, that is $\operatorname{int}(J_{\mathbf{u}i}) \cap \operatorname{int}(J_{\mathbf{u}j}) = \emptyset$ for $i \neq j$,

  (c) have diameters $\operatorname{diam}(J_{\mathbf{u}i}) = c_k \operatorname{diam}(J_{\mathbf{u}})$ for all $i = 1, \ldots, n_k$.

We call $F = \cap_{k=1}^{\infty} \cup_{\mathbf{u}\in D_k} J_{\mathbf{u}}$ a *homogeneous Moran set* if $\{J_{\mathbf{u}} | u \in D\}$ satisfies the Moran structure conditions (i) and (ii) above.

We adopt the following notation: we write $L_k = \prod_{i=1}^{k} c_i$ and $L_0 = 1$, and for each $\mathbf{u} \in D_k$ and each $m \in \mathbb{N} \cup \{0\}$ we write

$$D_{\mathbf{u},k+m} = \{\mathbf{v} \in D_{k+m} | v_1 \ldots v_k = \mathbf{u}\}$$

for the set of words of length $k+m$ which when truncated to the first k places are equal to the word $\mathbf{u}$. We also note that $\operatorname{card}(D_{\mathbf{u},k+m}) = \operatorname{card}(D_{k+m}) / \operatorname{card}(D_k)$.

More general Moran sets are considered in Li et al. [**12**] where the contraction ratios in (ii)(c) may differ between the k^{th}-level sets, that is $\operatorname{diam}(J_{\mathbf{u}i}) = c_{k,i} \operatorname{diam}(J_{\mathbf{u}})$ for all $i = 1, \ldots, n_k$, provided that the $c_{k,i}$ satisfy $\sum_{i=1}^{n_k} (c_{k,i})^d \leq 1$ for each $k \in \mathbb{N}$.

Before proving that homogeneous Moran sets are equi-homogeneous we first demonstrate that, due to the disjoint interiors property (ii)(b), the sets $J_{\mathbf{v}}$ are 'well separated' in the sense that there is a constant ω, independent of k, such that every ball of radius L_k intersects at most ω of the $J_{\mathbf{v}}$ with $\mathbf{v} \in D_k$.

LEMMA 2·7. *If $F \subset \mathbb{R}^d$ is a homogeneous Moran set then there exists a constant $\omega \in \mathbb{N}$ such that for all $x \in F$ and all $k \in \mathbb{N}$*

$$\operatorname{card}(\{v \in D_{k-1} | B_\delta(x) \cap J_v \neq \emptyset\}) \leq \omega$$

for all δ in the range $0 < \delta < L_{k-1}$.

Proof. Without loss of generality assume that $\operatorname{diam}(J) = 1$ and let $I \subset \operatorname{int}(J)$ be a ball of radius η for some $\eta < 1/2$. Let $I_{\mathbf{u}} = S_{\mathbf{u}}(I) \subset \operatorname{int}(J_{\mathbf{u}})$ and observe from the Moran structure conditions (i) and (ii)(c) that for $\mathbf{u} \in D_k$ the set $I_{\mathbf{u}}$ is a ball of radius ηL_k and that $\operatorname{diam}(J_{\mathbf{u}}) = L_k$.

Fix $x \in F$ and $k \in \mathbb{N}$ and let $\delta < L_{k-1}$. If $y \in B_\delta(x) \cap J_{\mathbf{v}}$ for some $\mathbf{v} \in D_{k-1}$ then $z \in I_{\mathbf{v}}$ implies that $|x - z| \leq |x - y| + |y - z| \leq \delta + \operatorname{diam}(J_{\mathbf{v}}) \leq 2L_{k-1}$, hence

$I_{\mathbf{v}} \subset B_{2L_{k-1}}(x)$. Consequently,

$$\begin{aligned}\operatorname{card}\left(\{v \in D_{k-1} | B_\delta(x) \cap J_v \neq \emptyset\}\right) &\leq \operatorname{card}\left(\left\{v \in D_{k-1} | I_{\mathbf{v}} \subset B_{2L_{k-1}}(x)\right\}\right) \\ &\leq \mu\left(B_{2L_{k-1}}(x)\right) / \mu\left(B_{\eta L_{k-1}}(0)\right) \\ &\leq (2/\eta)^d \mu(B_1(0))\end{aligned}$$

where μ is the Lebesgue measure on $\mathbb{R}^d$, as the balls $I_{\mathbf{v}}$ lie inside the interiors of the $J_{\mathbf{v}}$, which are pairwise disjoint by the Moran structure condition (ii)(b). Writing $\omega = \lceil (2/\eta)^d \mu(B_1(0)) \rceil$, where the ceiling function $\lceil x \rceil$ is the smallest integer not less than x, and noting that this constant is independent of x, δ and k completes the proof. □

In the remainder we adopt as our primary measure the geometric quantity $\mathcal{D}(F, \delta)$, which we recall is the minimum cover of sets of *diameter* δ, since it is convenient to cover a Moran set by the sets $J_{\mathbf{v}}$ whose diameter is known. This avoids the factor of 1/2 that would occur if we used covers by δ-balls

Theorem 2·8. *If $F \subset \mathbb{R}^d$ is a homogeneous Moran set such that $n_k \leq n^* < \infty$ for all $k \in \mathbb{N}$ then F is equi-homogeneous.*

Proof. Without loss of generality assume that $\operatorname{diam}(J) = 1$. First we demonstrate that for each $k \in \mathbb{N}$

$$\max_{\mathbf{v} \in D_k} \mathcal{D}(J_{\mathbf{v}} \cap F, \rho) \leq \omega n^* \min_{\mathbf{v} \in D_k} \mathcal{D}(J_{\mathbf{v}} \cap F, \rho/2) \qquad \text{for all } \rho > 0, \tag{2·9}$$

where ω is the constants from Lemma 2·7. Fix $k \in \mathbb{N}$ and $\mathbf{u}, \mathbf{v} \in D_k$. Fix $\rho \in (0, L_k)$ and let $m \in \mathbb{N}$ be such that $L_{k+m} \leq \rho < L_{k+m-1}$. Observe that from the Moran structure condition (ii)(a) the set $J_{\mathbf{u}} \cap F$ is contained in the union of the sets $J_{\mathbf{w}}$ with $\mathbf{w} \in D_{\mathbf{u},k+m}$, which have diameter L_{k+m}. Consequently,

$$\mathcal{D}(J_{\mathbf{u}} \cap F, \rho) \leq \mathcal{D}(J_{\mathbf{u}} \cap F, L_{k+m}) \leq \operatorname{card}(D_{\mathbf{u},k+m}) = \frac{\operatorname{card}(D_{k+m})}{\operatorname{card}(D_k)}. \tag{2·10}$$

Further, from the Moran structure condition (ii)(a)

$$J_{\mathbf{v}} \cap F = \bigcup_{\mathbf{w} \in D_{\mathbf{v},k+m-1}} J_{\mathbf{w}} \cap F$$

and each of the $J_{\mathbf{w}} \cap F$ is non-empty. As $\rho < L_{k+m-1}$ it follows from Lemma 2·7 that a ρ-ball intersects at most ω of the sets $J_{\mathbf{w}}$ with $\mathbf{w} \in D_{\mathbf{v},k+m-1}$. Consequently, at least $\operatorname{card}(D_{\mathbf{v},k+m-1})/\omega$ ρ-balls are required to intersect all of the $J_{\mathbf{w}}$ hence

$$\mathcal{N}(J_{\mathbf{v}} \cap F, \rho) \geq \operatorname{card}(D_{\mathbf{v},k+m-1})/\omega = \operatorname{card}(D_{\mathbf{v},k+m})/n_{k+m}\omega = \frac{\operatorname{card}(D_{k+m})}{n_{k+m}\omega \operatorname{card}(D_k)},$$

from which, with the geometric inequalities (1·2), we conclude that

$$\mathcal{D}(J_{\mathbf{v}} \cap F, \rho/2) \geq \mathcal{N}(J_{\mathbf{v}} \cap F, \rho) \geq \frac{\operatorname{card}(D_{k+m})}{n_{k+m}\omega \operatorname{card}(D_k)}. \tag{2·11}$$

It follows from the inequalities (2·10) and (2·11) and the bound $n_k \leq n^*$ that

$$\mathcal{D}(J_{\mathbf{u}} \cap F, \rho) \leq \omega n^* \mathcal{D}(J_{\mathbf{v}} \cap F, \rho/2) \tag{2·12}$$

for all $\rho \in (0, L_k)$. Further, if $\rho > L_k = \operatorname{diam}(J_{\mathbf{u}}) = \operatorname{diam}(J_{\mathbf{v}})$ then each of the sets $J_{\mathbf{u}} \cap F$ and $J_{\mathbf{v}} \cap F$ can be covered by a single set of diameter ρ, hence (2·12) holds for all $\rho > 0$. Finally, since $\mathbf{u}, \mathbf{v} \in D_k$ were chosen arbitrarily then (2·9) holds for all $k \in \mathbb{N}$.

Next, let $x, y \in F$, fix $\delta \in (0,1)$ and let $k \in \mathbb{N}$ be such that $L_k \leq \delta < L_{k-1}$. Now, $x \in J_{\mathbf{u}}$ for some $\mathbf{u} \in D_k$, and as $\operatorname{diam}(J_{\mathbf{u}}) = L_k \leq \delta$ it follows that $J_{\mathbf{u}} \subset B_\delta(x)$, hence

$$\min_{\mathbf{v} \in D_k} \mathcal{D}(J_{\mathbf{v}} \cap F, \rho) \leq \mathcal{D}(J_{\mathbf{u}} \cap F, \rho) \leq \mathcal{D}(B_\delta(x) \cap F, \rho) \tag{2·13}$$

for all $\rho > 0$. Next, as F is contained in the $(k-1)^{th}$-level sets $J_{\mathbf{v}}$ with $\mathbf{v} \in D_{k-1}$ it is clear that

$$B_\delta(y) \cap F \subset \bigcup_{\substack{v \in D_{k-1} \\ B_\delta(y) \cap J_{\mathbf{v}} \neq \emptyset}} J_{\mathbf{v}} \cap F \subset \bigcup_{\substack{\mathbf{v} \in D_{k-1} \\ B_\delta(y) \cap J_{\mathbf{v}} \neq \emptyset}} \bigcup_{i=1}^{n_k} J_{\mathbf{v}i} \cap F$$

where for each $\mathbf{v} \in D_{k-1}$ the sets $J_{\mathbf{v}i}$ are the k^{th}-level subsets of $J_{\mathbf{v}}$. Consequently,

$$\mathcal{D}(B_\delta(y) \cap F, \rho) \leq \operatorname{card}(\{\mathbf{v} \in D_{k-1} | B_\delta(y) \cap J_{\mathbf{v}} \neq \emptyset\}) n_k \max_{\mathbf{w} \in D_k} \mathcal{D}(J_{\mathbf{w}} \cap F, \rho)$$

for all $\rho > 0$. As $\delta < L_{k-1}$ it follows from Lemma 2·7 and the bound $n_k \leq n^*$ that

$$\mathcal{D}(B_\delta(y) \cap F, \rho) \leq \omega n^* \max_{\mathbf{w} \in D_k} \mathcal{D}(J_{\mathbf{w}} \cap F, \rho)$$

which, by (2·9) and (2·13),

$$\begin{aligned} &\leq (\omega n^*)^2 \min_{\mathbf{v} \in D_k} \mathcal{D}(J_{\mathbf{v}} \cap F, \rho/2) \\ &\leq (\omega n^*)^2 \mathcal{D}(B_\delta(x) \cap F, \rho/2) \end{aligned}$$

for all $\rho > 0$ and all δ in the range $L_k \leq \delta < L_{k-1}$, hence

$$\mathcal{D}(B_\delta(y) \cap F, \rho) \leq (\omega n^*)^2 \mathcal{D}(B_\delta(x) \cap F, \rho/2)$$

for all $\rho > 0$ and $\delta \in (0,1)$. Further, if $\delta > 1$ then $B_\delta(x) \cap F = F$ as $\operatorname{diam}(F) \leq \operatorname{diam}(J) = 1$, hence

$$\mathcal{D}(B_\delta(y) \cap F, \rho) = \mathcal{D}(F, \rho) \leq \mathcal{D}(F, \rho/2) = \mathcal{D}(B_\delta(x) \cap F, \rho/2)$$

as $\mathcal{D}(F, \cdot)$ is a non-increasing function. We conclude that

$$\mathcal{D}(B_\delta(y) \cap F, \rho) \leq (\omega n^*)^2 \mathcal{D}(B_\delta(x) \cap F, \rho/2)$$

for all $\rho > 0$ and $\delta > 0$. As $x, y \in F$ were chosen arbitrarily it follows that

$$\sup_{y \in F} \mathcal{D}(B_\delta(y) \cap F, \rho) \leq (\omega n^*)^2 \inf_{x \in F} \mathcal{D}(B_\delta(x) \cap F, \rho/2)$$

for all $\delta > 0$ and $\rho > 0$ so F is equi-homogeneous. $\square$

3. *Generalised Cantor Sets*

A generalised Cantor set is a variation of the well known Cantor middle third set that permits the proportion removed from each interval to vary throughout the iterative process. Formally, a generalised Cantor set is a homogeneous Moran set with initial set $J = [0,1]$, a sequence of contraction ratios $\{c_k\}_{k \in \mathbb{N}}$ with $c_k \in (0, 1/2)$, and $n_k = 2$ for all $k \in \mathbb{N}$. The intermediary sets $J_{\mathbf{u}}$ are inductively defined as follows: $J_\emptyset = [0,1]$ and for all $k \in \mathbb{N}$ and $\mathbf{u} \in D_{k-1}$

- $J_{\mathbf{u}1}$ is the interval of length $c_k \operatorname{diam}(J_{\mathbf{u}})$ sharing a common left end-point with the interval $J_{\mathbf{u}}$, and

- $J_{\mathbf{u}2}$ is the interval of length c_k diam $(J_{\mathbf{u}})$ sharing a common right end-point with the interval $J_{\mathbf{u}}$.

It is clear from the construction that the k^{th}-level intervals are obtained by removing the middle $1-2c_k$ proportion from $(k-1)^{th}$-level intervals. It follows that the k^{th}-level sets are disjoint from which it follows that the collection of sets $\{J_{\mathbf{u}} | \mathbf{u} \in D\}$ satisfies the Moran structure conditions so the set $C = \cap_{k=1}^{\infty} \cup_{\mathbf{u} \in D_k} J_{\mathbf{u}}$ is a homogeneous Moran set. The familiar Cantor middle third set can be obtained by setting $c_k = 1/3$ for all $k \in \mathbb{N}$.

It follows from Theorem 2·8 that generalised Cantor sets are equi-homogeneous and we will use this fact to determine the Assouad dimension of these sets. First, we calculate the box-counting dimensions of generalised Cantor sets, which can differ from the Assouad dimension as the examples in the next section will show.

Feng et al. [**6**] calculated the upper box-counting dimension (therein referred to as the upper Bouligand dimension) of a class of sets, including the generalised Cantor sets, as an intermediary result to find their packing dimension. Here we provide a direct calculation of both the upper and the lower box-counting dimensions of generalised Cantor sets.

It is not difficult to determine that for δ in the range $L_n \leq \delta < L_{n-1}$ the minimum number of sets of diameter δ required to cover C satisfies

$$2^{n-1} \leq \mathcal{D}(C, \delta) \leq 2^n \tag{3·1}$$

(see, for example, Robinson & Sharples [**20**].) From this bound we can determine the upper and lower box-counting dimensions of C from the sequence $\{c_i\}_{i\in\mathbb{N}}$.

Lemma 3·1. *Let C be the generalised Cantor with contraction ratios $\{c_i\}_{i\in\mathbb{N}}$. The lower and upper box-counting dimensions of C satisfy*

$$\dim_{\mathrm{LB}} C = \liminf_{n\to\infty} \frac{n \log 2}{-\sum_{i=1}^{n} \log c_i} \tag{3·2}$$

and

$$\dim_{\mathrm{B}} C = \limsup_{n\to\infty} \frac{n \log 2}{-\sum_{i=1}^{n} \log c_i}. \tag{3·3}$$

Proof. Fix δ in the range $0 < \delta < 1$ and let $n \in \mathbb{N}$ be such that $L_n \leq \delta < L_{n-1}$. The cover estimates (3·1) yield

$$\frac{(n-1)\log 2}{-\log L_n} \leq \frac{\log \mathcal{D}(C,\delta)}{-\log \delta} \leq \frac{n \log 2}{-\log L_{n-1}}$$

from which we derive

$$\frac{n\log 2}{-\log L_n} - \frac{\log 2}{-\log L_n} \leq \frac{\log \mathcal{D}(C,\delta)}{-\log \delta} \leq \frac{(n-1)\log 2}{-\log L_{n-1}} + \frac{\log 2}{-\log L_{n-1}}. \tag{3·4}$$

Taking limits as $\delta \to 0$, it is clear that $n \to \infty$ and $1/-\log L_n \to 0$ so taking the limit inferior of (3·4) we obtain

$$\liminf_{n\to\infty} \frac{n\log 2}{-\log L_n} \leq \liminf_{\delta\to 0+} \frac{\log \mathcal{D}(C,\delta)}{-\log \delta} \leq \liminf_{n\to\infty} \frac{(n-1)\log 2}{-\log L_{n-1}} \tag{3·5}$$

and as the upper and lower bounds of (3·5) are equal we conclude that

$$\liminf_{\delta\to 0+} \frac{\log \mathcal{D}(C,\delta)}{-\log \delta} = \liminf_{n\to\infty} \frac{n\log 2}{-\log L_n} = \liminf_{n\to\infty} \frac{n \log 2}{-\sum_{i=1}^{n}\log c_i},$$

which is precisely (3·2). The upper box-counting dimension equality (3·3) follows similarly after taking the limit superior of (3·4). □

This relationship is particularly pleasing as $\frac{n}{\sum_{i=1}^{n}\log c_i} = \frac{1}{\log a_n}$ where a_n is nothing more than the geometric mean of the partial sequence $c_1, \ldots, c_n$.

In Corollary 1 of Li et al. [**12**] the authors provide an expression for the Assouad dimension of homogeneous Moran sets, which include the generalised Cantor sets, provided that $\inf_{k\in\mathbb{N}} c_k > 0$. Here we calculate the Assouad dimension for all generalised Cantor sets with no restriction on the contraction ratios c_k.

LEMMA 3·2. *Let C be the generalised Cantor set with contraction ratios $\{c_i\}_{i\in\mathbb{N}}$. The Assouad dimension of C satisfies*

$$\dim_{\mathrm{A}} C = \limsup_{m\to\infty}\left(\sup_{n\in\mathbb{N}} \frac{m\log 2}{-\sum_{i=n+1}^{n+m}\log c_i}\right). \tag{3·6}$$

Proof. As C is equi-homogeneous from Theorem 2·8, it follows from Corollary 2·4 and the geometric inequalities (1·2) that for all $\delta_0 > 0$ there exist constants $M \geq 1$ and $\lambda_1, \lambda_2 > 0$ with $\lambda_2 \leq \lambda_1$ such that

$$\sup_{x\in C}\mathcal{D}\left(B_{\delta/8\lambda_1}(x)\cap C, 4\rho/\lambda_2\right) \leq M\frac{\mathcal{D}(C,\rho)}{\mathcal{D}(C,\delta)} \tag{3·7}$$

and

$$\sup_{x\in C}\mathcal{D}(B_{2\delta}(x)\cap C, \rho/4) \geq \frac{\mathcal{D}(C,\rho)}{\mathcal{D}(C,\delta)} \tag{3·8}$$

for all δ, ρ with $0 < \rho < \delta < \delta_0$.

For brevity we write $\beta = \limsup_{m\to\infty}(\sup_{n\in\mathbb{N}} m\log 2/(-\sum_{i=n+1}^{n+m}\log c_i))$ and observe that this limit exists and is finite as

$$\frac{m\log 2}{-\sum_{i=n+1}^{n+m}\log c_i} \leq \frac{m\log 2}{-\sum_{i=n+1}^{n+m}\log 1/2} = 1 \qquad \text{for all } n\in\mathbb{N} \text{ and } m\in\mathbb{N}\cup\{0\}.$$

Fix $\varepsilon > 0$ and let $m_\varepsilon \geq 2$ be sufficiently large that $\sup_{n\in\mathbb{N}} \frac{m\log 2}{\sum_{i=n+1}^{n+m}\log c_i} < \beta + \varepsilon$ for all $m > m_\varepsilon$.

Let δ, ρ satisfy $0 < \rho < \delta < \delta_0$, and let $n \in \mathbb{N}$ and $m \in \mathbb{N}\cup\{0\}$ be such that $L_n \leq \delta < L_{n-1}$ and $L_{n+m} \leq \rho < L_{n+m-1}$. From (3·7) and the cover estimates (3·1) it follows that

$$\sup_{x\in C}\mathcal{D}\left(B_{\delta/8\lambda_1}(x)\cap C, 4\rho/\lambda_2\right) \leq M\frac{2^{n+m}}{2^{n-1}} = M2^{m-1}. \tag{3·9}$$

Now, if $m - 1 \leq m_\varepsilon$ then from (3·9)

$$\sup_{x\in C}\mathcal{D}\left(B_{\delta/8\lambda_1}(x)\cap C, 4\rho/\lambda_2\right) \leq M2^{m_\varepsilon} \leq M2^{m_\varepsilon}(\delta/\rho)^{\beta+\varepsilon} \tag{3·10}$$

as $\delta/\rho > 1$. Alternatively, if $m - 1 > m_\varepsilon$ then from (3·9)

$$\begin{aligned}\sup_{x\in C}\mathcal{D}\left(B_{\delta/8\lambda_1}(x)\cap C, 4\rho/\lambda_2\right) \leq M2^{m-1} &= M(\delta/\rho)^{\frac{(m-1)\log 2}{\log(\delta/\rho)}} \\ &\leq M(\delta/\rho)^{\frac{(m-1)\log 2}{-\sum_{i=n+1}^{n+m-1}\log c_i}},\end{aligned}$$

as $\delta/\rho > L_n/L_{n+m-1} = \prod_{i=n+1}^{n+m-1} 1/c_i$ for $m \geq 2$,

$$\begin{aligned}&\leq M(\delta/\rho)^{\sup_{n\in\mathbb{N}}\frac{(m-1)\log 2}{-\sum_{i=n+1}^{n+m-1}\log c_i}} \\ &\leq M(\delta/\rho)^{\beta+\varepsilon}.\end{aligned} \tag{3·11}$$

Combining (3·10) and (3·11) we obtain

$$\sup_{x\in C}\mathcal{D}\left(B_{\delta/8\lambda_1}(x)\cap C, 4\rho/\lambda_2\right)\le M2^{m_\varepsilon}\left(\delta/\rho\right)^{\beta+\varepsilon}$$

for all δ,ρ satisfying $0<\rho<\delta<\delta_0$, from which it follows that C is $(\beta+\varepsilon)$-homogeneous. As $\varepsilon>0$ was arbitrary we conclude that $\dim_{\mathrm{A}} C\le\beta$.

Next, we suppose for a contradiction that $\dim_{\mathrm{A}} C<\beta$, so there exists an $\varepsilon>0$ and a constant $M>0$ such that

$$\mathcal{D}(B_\delta(x)\cap C,\rho)\le M\left(\delta/\rho\right)^{\beta-\varepsilon}. \tag{3·12}$$

From the definition of β there exists an increasing sequence $m_j\in\mathbb{N}$ (for convenience we assume without loss of generality that $m_j\ge 2$) such that

$$\sup_{n\in\mathbb{N}}\frac{m_j\log 2}{-\sum_{i=n+1}^{n+m_j}\log c_i}>\beta-1/j\qquad\text{for all } j\in\mathbb{N}$$

and for each $j\in\mathbb{N}$ there exists an $n_j\in\mathbb{N}$ such that

$$\frac{m_j\log 2}{-\sum_{i=n_j+1}^{n_j+m_j}\log c_i}>\sup_{n\in\mathbb{N}}\frac{m_j\log 2}{-\sum_{i=n+1}^{n+m_j}\log c_i}-\varepsilon/2>\beta-1/j-\varepsilon/2.$$

Now, let $\delta_j=L_{n_j}$ and $\rho_j=L_{n_j+m_j}$ so from (3·8) and the cover estimates (3·1) it follows that

$$\begin{aligned}\sup_{x\in C}\mathcal{D}(B_{2\delta_j}(x)\cap C,\rho_j/4)\ge\frac{2^{n_j+m_j-1}}{2^{n_j}}=2^{-1}2^{m_j}&=2^{-1}\left(\delta_j/\rho_j\right)^{\frac{m_j\log 2}{\log(\delta_j/\rho_j)}}\\&=2^{-1}\left(\delta_j/\rho_j\right)^{\frac{m_j\log 2}{\log\left(L_{n_j}/L_{n_j+m_j}\right)}}\\&=2^{-1}\left(\delta_j/\rho_j\right)^{\frac{m_j\log 2}{-\sum_{i=n_j+1}^{n_j+m_j}\log c_i}}\\&\ge 2^{-1}\left(\delta_j/\rho_j\right)^{\beta-1/j-\varepsilon/2}\end{aligned}$$

for all $j\in\mathbb{N}$. It follows from (3·12) that

$$2^{-1}\left(\delta_j/\rho_j\right)^{\beta-1/j-\varepsilon/2}\le M\left(\delta_j/\rho_j\right)^{\beta-\varepsilon}\qquad\text{for all } j\in\mathbb{N}$$

and by rearranging and recalling that $\delta_j/\rho_j=L_{n_j}/L_{n_j+m_j}=\prod_{i=n_j+1}^{n_j+m_j}1/c_i\ge 2^{m_j}$ that

$$2^{-1}2^{m_j(\varepsilon/2-1/j)}=2^{-1}\left(\delta_j/\rho_j\right)^{\varepsilon/2-1/j}\le M\qquad\text{for all } j\in\mathbb{N}.$$

Finally, as $1/j<\varepsilon/4$ for all j greater than some $J_\varepsilon>0$ it follows that

$$2^{-1}2^{m_j\varepsilon/4}\le M\qquad\text{for all } j>J_\varepsilon,$$

which is a contradiction as the m_j are increasing, hence $2^{m_j\varepsilon/4}$ is unbounded. We conclude that the generalised Cantor set C is not $(\beta-\varepsilon)$-homogeneous for any $\varepsilon>0$, hence $\dim_{\mathrm{A}} C=\beta$, proving the lemma. □

4. *Strict inequality in the two product formulae*

In this section we provide a method for constructing two generalised Cantor sets C and D so that the Assouad dimensions of these sets and their product satisfy

$$\dim_{\mathrm{A}} C=\dim_{\mathrm{A}} D=\dim_{\mathrm{A}}(C\times D)=\alpha$$

for $\alpha \in (0,1)$. In particular for these sets the Assouad dimension product inequality (1·9) is strict and maximal in the sense that the sum $\dim_{\mathrm{A}} C + \dim_{\mathrm{A}} D$ takes the maximal value $2\dim_{\mathrm{A}}(C \times D)$.

This task is significantly simplified using the results of the previous sections that relate the Assouad dimension to the more manageable box-counting dimensions. In essence we construct these sets so that the significant length-scales are common to both sets, which is similar in approach to the compatible generalised Cantor sets of Robinson & Sharples [**20**].

Let $q \in (0, \frac{1}{2})$ and let $a = \{a_i\}$ be a sequence of positive integers. We define two generalised Cantor sets C and D via the respective sequences of contraction ratios $\{c_i\}_{i\in\mathbb{N}}$ and $\{d_i\}_{i\in\mathbb{N}}$ defined by

$$c_i := \begin{cases} q^{a_{2k}+1} & i = n_k \quad \text{for some } k \in \mathbb{N} \\ q & \text{otherwise} \end{cases}$$

$$d_i := \begin{cases} q^{a_{2k+1}+1} & i = m_k \quad \text{for some } k \in \mathbb{N} \\ q & \text{otherwise,} \end{cases}$$

where $n_k = \sum_{j=1}^{k} a_{2j-1}$ and $m_k = a_1 + \sum_{j=1}^{k} a_{2j}$. For brevity we say that the pair of sets (C, D) is generated by (q, a), and we denote the partial sum $s_k = \sum_{i=1}^{k} a_i$. Essentially, the sequences of contraction ratios c_i and d_i are chosen so that, when δ is restricted to the range $[q^{s_{k+1}}, q^{s_k}]$, one of the functions $\mathcal{D}(C,\delta)$ or $\mathcal{D}(D,\delta)$ scales like $\delta^{-\log 2/\log q}$ while the other is essentially constant, and such that these roles alternate as k increases. While the growth of the individual functions $\mathcal{D}(C,\delta)$ and $\mathcal{D}(D,\delta)$ fluctuates with δ, the product $\mathcal{D}(C,\delta)\mathcal{D}(D,\delta)$ scales like $\delta^{-\log 2/\log q}$ for all δ.

THEOREM 4·1. *Let the pair of generalised Cantor sets C and D be generated by (q, a). For all $\delta_0 > 0$ there exists a constant $\eta > 0$ such that*

$$\eta^{-1}\delta^{-\frac{\log 2}{\log q}} \le \mathcal{D}(C \times D, \delta) \le \eta\, \delta^{-\frac{\log 2}{\log q}} \qquad \textit{for all } 0 < \delta < \delta_0, \tag{4·1}$$

so that in particular

$$\dim_{\mathrm{LB}}(C \times D) = \dim_{\mathrm{B}}(C \times D) = \dim_{A}(C \times D) = -\log 2/\log q.$$

Proof. Using the terminology of the previous section, the n^{th}-level sets in the construction of C and D consist of 2^n intervals of length $L_n := \prod_{i=1}^{n} c_i$ and $M_n := \prod_{i=1}^{n} d_i$ respectively.

We first consider the generalised Cantor set C. For $n \in \mathbb{N}$ in the range $n_k \le n < n_{k+1}$ all except k of the $c_1, \ldots, c_n$ are equal to q, so

$$L_n = q^{n-k} \prod_{i=1}^{k} q^{a_{2i}+1} = q^n \prod_{i=1}^{k} q^{a_{2i}}.$$

Taking logarithms for clarity, we derive

$$\begin{aligned} \frac{\log L_n}{\log q} &= n - k + \sum_{i=1}^{k} (a_{2i} + 1) = n + \sum_{i=1}^{k} a_{2i} \\ &= n - \sum_{i=1}^{k} a_{2i-1} + \sum_{i=1}^{k} a_{2i-1} + \sum_{i=1}^{k} a_{2i} = n - n_k + s_{2k}, \end{aligned}$$

so

$$L_n = q^{n-n_k+s_{2k}} \qquad n_k \le n < n_{k+1}, \tag{4·2}$$

and, in particular, $L_{n_k} = q^{s_{2k}}$ and $L_{n_{k+1}-1} = q^{s_{2k+1}-1}$. Observe that for $n_k \le n < n_{k+1}$ the length L_n has range $\left[L_{n_{k+1}-1}, L_{n_k}\right] = \left[q^{s_{2k+1}-1}, q^{s_{2k}}\right]$, so inverting the relationship (4·2), we derive

$$q^j = L_{n_k+j-s_{2k}}, \qquad s_{2k} \le j \le s_{2k+1}-1,$$

so, from the cover estimates (3·1),

$$2^{n_k+j-s_{2k}-1} \le \mathcal{D}(C, q^j) \le 2^{n_k+j-s_{2k}} \qquad s_{2k} \le j \le, s_{2k+1}-1. \tag{4·3}$$

Further, as $k \in \mathbb{N}$ was arbitrary in the derivation of (4·2) it follows that $L_{n_{k+1}} = q^{s_{2k+2}}$ and $L_{n_{k+1}-1} = q^{s_{2k+1}-1}$, so

$$L_{n_{k+1}} \le q^j \le L_{n_{k+1}-1} \qquad s_{2k+1}-1 \le j \le s_{2k+2},$$

from which, with the cover estimates (3·1), we conclude that

$$2^{n_{k+1}-2} \le \mathcal{D}(C, q^j) \le 2^{n_{k+1}} \qquad s_{2k+1}-1 \le j \le, s_{2k+2}. \tag{4·4}$$

A very similar argument shows that for the set D the bounds

$$2^{m_k+j-s_{2k+1}-1} \le \mathcal{D}(D, q^j) \le 2^{m_k+j-s_{2k+1}} \qquad s_{2k+1} \le j \le s_{2k+2}-1. \tag{4·5}$$

and

$$2^{m_k-2} \le \mathcal{D}(D, q^j) \le 2^{m_k} \qquad s_{2k}-1 \le j \le s_{2k+1}. \tag{4·6}$$

hold.

Now, taking the product of (4·4) and (4·5) we obtain

$$2^{n_{k+1}+m_k+j-s_{2k+1}-3} \le \mathcal{D}(C, q^j)\mathcal{D}(D, q^j) \le 2^{n_{k+1}+m_k+j-s_{2k+1}} \tag{4·7}$$

for $s_{2k+1} \le j \le s_{2k+2}-1$, and multiplying (4·3) with (4·6) yields

$$2^{n_k+m_k+j-s_{2k}-3} \le \mathcal{D}(C, q^j)\mathcal{D}(D, q^j) \le 2^{n_k+m_k+j-s_{2k}} \tag{4·8}$$

for $s_{2k} \le j \le s_{2k+1}-1$. Finally, since $n_k + m_k = s_{2k} + a_1$ and $n_{k+1} + m_k = s_{2k+1} + a_1$, the bounds (4·7) and (4·8) are precisely

$$2^{j+a_1-3} \le \mathcal{D}(C, q^j)\mathcal{D}(D, q^j) \le 2^{j+a_1} \tag{4·9}$$

for $s_{2k} \le j \le s_{2k+2}-1$ and, as $k \in \mathbb{N}$ was arbitrary, we see that (4·9) holds for all $j \ge s_2$.

Fix $\delta \in (0, q^{s_2})$ and let $j \ge s_2$ be such that $q^{j+1} \le \delta < q^j$. As $\mathcal{D}(C, \cdot)$ and $\mathcal{D}(D, \cdot)$ are non-increasing it follows that

$$\mathcal{D}(C, q^j)\mathcal{D}(D, q^j) \le \mathcal{D}(C, \delta)\mathcal{D}(D, \delta) \le \mathcal{D}(C, q^{j+1})\mathcal{D}(D, q^{j+1}),$$

which from (4·9) implies that

$$2^{a_1-4}\left(q^{j+1}\right)^{\frac{\log 2}{\log q}} = 2^{j+a_1-3} \le \mathcal{D}(C, \delta)\mathcal{D}(D, \delta) \le 2^{j+1+a_1} = 2^{a_1+1}\left(q^j\right)^{\frac{\log 2}{\log q}}.$$

As $\log 2/\log q < 0$ it follows that

$$2^{a_1-4}\delta^{\frac{\log 2}{\log q}} \leq \mathcal{D}(C,\delta)\mathcal{D}(D,\delta) \leq 2^{a_1+1}\delta^{\frac{\log 2}{\log q}} \tag{4·10}$$

for all $0 < \delta < q^{s_2}$.

Finally, recall from the product inequality (1·12) and the geometric relationships (1·2) that for all $\delta > 0$

$$\mathcal{D}(C,8\delta)\mathcal{D}(D,8\delta) \leq \mathcal{D}(C\times D,\delta) \leq \mathcal{D}(C,\delta/2\sqrt{2})\mathcal{D}(D,\delta/2\sqrt{2}),$$

whence

$$\eta^{-1}\delta^{\frac{\log 2}{\log q}} \leq \mathcal{D}(C\times D,\delta) \leq \eta\delta^{\frac{\log 2}{\log q}} \qquad \text{for all } 0 < \delta < q^{s_2}. \tag{4·11}$$

If $\delta_0 < q^{s_2}$ then (4·11) implies (4·1), otherwise observe that for δ in the range $q^{s_2} \leq \delta \leq \delta_0$ trivially

$$\mathcal{D}\left(C\times D,\delta_0\right)\left(q^{s_2}\right)^{-\frac{\log 2}{\log q}}\delta^{\frac{\log 2}{\log q}} \leq \mathcal{D}\left(C\times D,\delta\right) \leq \mathcal{D}\left(C\times D,q^{s_2}\right)\delta_0^{-\frac{\log 2}{\log q}}\delta^{\frac{\log 2}{\log q}}$$

which, together with (4·11), yields (4·1).

This immediately shows that the upper and lower box-counting dimensions coincide, are attained, and are equal to $-\log 2/\log q$. The same expression for the Assouad dimension then follows using Theorem 2·5 and the fact that the product set $C \times D$ is equi-homogeneous, being the product of two equi-homogeneous sets C and D (Lemma 2·6 and Theorem 2·8). □

THEOREM 4·2. *Let the pair of generalised Cantor sets C and D be generated by (q,a). The upper box-counting dimensions of C and D are given by*

$$\dim_{\mathrm{B}} C = -\left(\frac{\log 2}{\log q}\right)\limsup_{k\to\infty}\frac{\sum_{j=1}^{k} a_{2j-1}}{\sum_{j=1}^{2k-1} a_i} \tag{4·12}$$

and $$\dim_{\mathrm{B}} D = -\left(\frac{\log 2}{\log q}\right)\limsup_{k\to\infty}\frac{\sum_{j=1}^{k} a_{2j}}{\sum_{i=1}^{2k} a_i}$$

respectively.

Proof. Recall from Lemma 3·1 that the upper-box counting dimensions of C and D are given by

$$\dim_{\mathrm{B}} C = \limsup_{n\to\infty}\frac{n\log 2}{-\log L_n} \qquad \text{and} \qquad \dim_{\mathrm{B}} D = \limsup_{n\to\infty}\frac{n\log 2}{-\log M_n}.$$

We first consider the generalised Cantor set C. For $n \in \mathbb{N}$ in the range $n_k \leq n < n_{k+1}$ we obtain from (4·2)

$$\begin{aligned}\frac{n\log 2}{-\log L_n} = \frac{n\log 2}{-\left(n-n_k+s_{2k}\right)\log q} &\leq -\left(\frac{\log 2}{\log q}\right)\frac{n_{k+1}}{s_{2k+1}}\\ &= -\left(\frac{\log 2}{\log q}\right)\frac{\sum_{i=1}^{k+1} a_{2j-1}}{\sum_{i=1}^{2k+1} a_i},\end{aligned}$$

where we have used the fact that $n/(a+n)$ is increasing in n for $a > 0$. Taking the limit superior as n (and hence k) tend to infinity we conclude that

$$\dim_{\mathrm{B}} C \leq -\left(\frac{\log 2}{\log q}\right)\limsup_{k\to\infty}\frac{\sum_{i=1}^{k+1} a_{2j-1}}{\sum_{i=1}^{2k+1} a_i},$$

which is the upper bound in (4·12). To establish the lower bound we consider the subsequence $n_{k+1}-1$ and recall from (4·2) that $L_{n_{k+1}-1} = q^{s_{2k+1}-1}$. Consequently,

$$\frac{(n_{k+1}-1)\log 2}{-\log L_{n_{k+1}-1}} = \frac{(n_{k+1}-1)\log 2}{-(s_{2k+1}-1)\log q} = -\left(\frac{\log 2}{\log q}\right)\frac{\sum_{j=1}^{k+1} a_{2j-1}-1}{\sum_{i=1}^{2k+1} a_i - 1},$$

so

$$\begin{aligned}\dim_{\mathrm{B}} C = \limsup_{n\to\infty}\frac{n\log 2}{-\log L_n} &\geq \limsup_{k\to\infty}\frac{(n_{k+1}-1)\log 2}{-\log L_{n_{k+1}-1}} \\ &= -\left(\frac{\log 2}{\log q}\right)\limsup_{k\to\infty}\frac{\sum_{j=1}^{k+1} a_{2j-1}}{\sum_{i=1}^{2k+1} a_i}.\end{aligned}$$

Since these upper and lower bounds coincide we obtain the equality in (4·12).

The argument for D follows similar lines. □

The Assouad dimension dominates the upper box-counting dimension, so the above theorem provides lower bounds for the Assouad dimension of the sets C and D. However, using the results of the previous section, we can precisely determine the Assouad dimension of the sets C and D provided that the odd and even terms of the sequence $\{a_i\}_{i\in\mathbb{N}}$ respectively are unbounded. Essentially, for all $k \in \mathbb{N}$ the contraction ratios $\{c_i\}_{i\in\mathbb{N}}$ contains a string of $a_{2k+1}-1$ consecutive ratios c_i equal to q. This string corresponds to a range of length-scales for which $\sup_{x\in C}\mathcal{D}(B_\delta(x)\cap C, \rho)$ scales like $(\delta/\rho)^{-\log 2/\log q}$. If the a_{2k+1} are unbounded then this scaling holds for δ and ρ arbitrarily close to zero with the ratio δ/ρ arbitrarily large, from which it follows that $\dim_{\mathrm{A}} C$ is at least $(-\log 2/\log q)$. This approach using strings of increasing length is similar to that used in Fraser et al. [**8**] to find lower bounds on the Assouad dimension of attractors of random iterated function systems.

Lemma 4·3. *If the generalised Cantor sets* (C, D) *are generated by* (q, a) *then*

$$\begin{aligned} &\sup\{a_{2i-i}\} = \infty \quad \textit{implies} \quad \dim_{\mathrm{A}} C = -\tfrac{\log 2}{\log q}, \\ \textit{and} \quad &\sup\{a_{2i}\} = \infty \quad \textit{implies} \quad \dim_{\mathrm{A}} D = -\tfrac{\log 2}{\log q}.\end{aligned}$$

Proof. Observe that

$$\frac{m\log 2}{-\sum_{i=n+1}^{n+m}\log c_i} \leq \frac{m\log 2}{-m\log q} = -\frac{\log 2}{\log q}$$

for all $n \in \mathbb{N}$ and $m \in \mathbb{N}\cup\{0\}$, as $c_i \leq q$ for all $i \in \mathbb{N}$. It follows from Lemma 3·2 that $\dim_{\mathrm{A}} C \leq \log 2/\log q$.

Next, observe that $\sum_{i=n_k+1}^{n_k+a_{2k+1}-1}\log c_i = (a_{2k+1}-1)\log q$ for each $k \in \mathbb{N}$ as $c_i = q$ for all $i = n_k+1, \ldots, n_{k+1}-1 = n_k + a_{2k+1} - 1$. Consequently,

$$\sup_{n\in\mathbb{N}}\frac{(a_{2k+1}-1)\log 2}{-\sum_{i=n+1}^{n+a_{2k+1}-1}\log c_i} \geq \frac{(a_{2k+1}-1)\log 2}{-(a_{2k+1}-1)\log q} = -\frac{\log 2}{\log q}.$$

Assuming that the sequence a_{2k+1} is unbounded it follows that

$$\limsup_{m\to\infty}\sup_{n\in\mathbb{N}}\frac{m\log 2}{-\sum_{i=n+1}^{n+m}\log c_i} \geq -\frac{\log 2}{\log q}$$

which, from Lemma 3·2, is precisely that $\dim_{\mathrm{A}} C \geq -\log 2/\log q$, so we conclude that

$\dim_A C = -\log 2/\log q$ if the sequence a_{2k+1} is unbounded. The argument for the set D follows similar lines. $\square$

In summary we have constructed generalised Cantor sets C and D such that

$$\dim_B C = -\frac{\log 2}{\log q} \limsup_{k\to\infty} \frac{\sum_{i=1}^{k+1} a_{2j-1}}{\sum_{i=1}^{2k+1} a_i},$$

$$\dim_B D = -\frac{\log 2}{\log q} \limsup_{k\to\infty} \frac{\sum_{j=1}^{k+1} a_{2j}}{\sum_{i=1}^{2k+2} a_i},$$

$$\dim_A (C \times D) = \dim_B (C \times D) = -\frac{\log 2}{\log q},$$

$$\dim_A C = -\frac{\log 2}{\log q} \qquad \text{if } \{a_{2i-1}\} \text{ is unbounded,}$$

$$\text{and} \qquad \dim_A D = -\frac{\log 2}{\log q} \qquad \text{if } \{a_{2i}\} \text{ is unbounded.}$$

By choosing the $\{a_i\}$ appropriately we can now produce generalised Cantor sets C and D such that

$$\dim_A C = \dim_A D = \dim_A (C \times D) = \dim_B (C \times D) = \dim_{LB} (C \times D),$$

where the box-counting dimensions of these sets take arbitrary values satisfying the product formula

$$\dim_B C, \dim_B D \le \dim_B (C \times D) \le \dim_B C + \dim_B D,$$

and

$$0 < \dim_B C, \dim_B D < \dim_B (C \times D) < 1.$$

In particular the Assouad dimension of the product satisfies

$$\dim_A (C \times D) < \dim_A C + \dim_A D = 2 \dim_A (C \times D)$$

so there is a strict inequality in the Assouad dimension product formula (1·9). Further, these sets give extreme examples of strict inequality in the product formula as $\dim_A F + \dim_A G \le 2 \dim_A (F \times G)$ for arbitrary sets F and G.

LEMMA 4·4. *Let $\alpha, \beta \in (0,1)$. There exist generalised Cantor sets C and D such that*

$$\dim_{LB} C = \dim_B C = \alpha\beta, \qquad \dim_{LB} D = \dim_B D = \alpha (1-\beta),$$

and

$$\dim_{LB} (C \times D) = \dim_B (C \times D) = \dim_A (C \times D) = \dim_A C = \dim_A D = \alpha.$$

Proof. Define the sequence $a = \{a_i\}$ by $a_{2k-1} = \lceil \beta k \rceil$ and $a_{2k} = \lceil (1-\beta) k \rceil$ where the ceiling function $\lceil x \rceil$ is the smallest integer greater than or equal to x. Clearly $\beta, 1-\beta > 0$ so the a_i are positive integers. Let the pair of generalised Cantor sets C and D be generated by $\left(2^{-1/\alpha}, a\right)$, so immediately from Theorem 4·1 we obtain

$$\dim_{LB} (C \times D) = \dim_B (C \times D) = \dim_A (C \times D) = \alpha$$

as required. Further, as both the odd and even terms a_{2i-1} and a_{2i} are unbounded we obtain $\dim_A C = \dim_A D = \alpha$ from Lemma 4·3.

Next, observe that

$$\tfrac{1}{2}k\,(k+1)\,(1-\beta) \le \sum_{j=1}^{k} a_{2j} \le \tfrac{1}{2}k\,(k+1)\,(1-\beta) + k$$

$$\text{and} \qquad \tfrac{1}{2}k\,(k+1)\,\beta \le \sum_{j=1}^{k} a_{2j-1} \le \tfrac{1}{2}k\,(k+1)\,\beta + k.$$

Consequently,

$$\begin{aligned}\frac{\sum_{j=1}^{k} a_{2j-1}}{\sum_{j=1}^{2k-1} a_i} &\le \frac{\tfrac{1}{2}k\,(k+1)\,\beta + k}{\tfrac{1}{2}k\,(k-1)\,(1-\beta) + \tfrac{1}{2}k\,(k+1)\,\beta} \\ &= \frac{(k+1)\,\beta + 2}{k-1+2\beta} \to \beta\end{aligned}$$

as $k \to \infty$, while

$$\begin{aligned}\frac{\sum_{j=1}^{k} a_{2j-1}}{\sum_{j=1}^{2k-1} a_i} &\ge \frac{\tfrac{1}{2}k\,(k+1)\,\beta}{\tfrac{1}{2}k\,(k-1)\,(1-\beta) + k - 1 + \tfrac{1}{2}k\,(k+1)\,\beta + k} \\ &= \frac{k\,(k+1)\,\beta}{k^2 + k - 1 + 2k\beta} \to \beta\end{aligned}$$

as $k \to \infty$. It follows from Theorem 4·2 that $\dim_{\mathrm{B}} C = \alpha\beta$ as required, and from a similar argument we obtain $\dim_{\mathrm{B}} D = \alpha\,(1-\beta)$. Finally, observe that from the chain of product inequalities (1·10) we obtain

$$\dim_{\mathrm{LB}} C + \dim_{\mathrm{B}} D = \dim_{\mathrm{B}} C + \dim_{\mathrm{LB}} D = \dim_{\mathrm{B}}\,(C \times D)\,,$$

which implies that $\dim_{\mathrm{LB}} C = \alpha\beta$ and $\dim_{\mathrm{LB}} D = \alpha\,(1-\beta)$. $\square$

The previous lemma is a limiting case of the following more general construction, which gives independent control over the box-counting dimensions of C and D.

Lemma 4·5. *Let $\alpha, \beta, \gamma \in (0,1)$ be such that $\beta + \gamma > 1$. There exist generalised Cantor sets C and D such that*

$$\begin{aligned} \dim_{\mathrm{LB}} C &= \alpha\,(1-\gamma)\,, & \dim_{\mathrm{B}} C &= \alpha\beta, \\ \dim_{\mathrm{LB}} D &= \alpha\,(1-\beta)\,, & \dim_{\mathrm{B}} D &= \alpha\gamma, \end{aligned}$$

and

$$\dim_{\mathrm{LB}}\,(C \times D) = \dim_{\mathrm{B}}\,(C \times D) = \dim_{\mathrm{A}}\,(C \times D) = \dim_{\mathrm{A}} C = \dim_{\mathrm{A}} D = \alpha.$$

Proof. We first observe that $\frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma} > 0$ and that

$$\frac{\gamma\beta}{(1-\gamma)\,(1-\beta)} > 1 \tag{4·13}$$

follows from $\beta + \gamma > 1$.

Now recursively define the sequence $\{a_i\}$ by

$$\begin{aligned} a_1 &= 1, \\ a_2 &= \left\lceil \tfrac{\gamma}{1-\gamma} \right\rceil + 1, \\ a_{2k+1} &= \left\lceil \tfrac{\beta}{1-\beta} e_k - o_k \right\rceil + 1, && (4\cdot14) \\ \text{and} \qquad a_{2k+2} &= \left\lceil \tfrac{\gamma}{1-\gamma} o_{k+1} - e_k \right\rceil + 1 && (4\cdot15) \end{aligned}$$

for $k \in \mathbb{N}$, where $o_k = \sum_{j=1}^{k} a_{2j-1}$ and $e_k = \sum_{j=1}^{k} a_{2j}$ are the sums of the odd and of the even terms of a_i respectively. Observe that

$$\begin{aligned} a_{2k+2} \geq \tfrac{\gamma}{1-\gamma} o_{k+1} - e_k + 1 &= \tfrac{\gamma}{1-\gamma} \left(a_{2k+1} + o_k\right) - e_k + 1 \\ &\geq \tfrac{\gamma}{1-\gamma} \left(\tfrac{\beta}{1-\beta} e_k + 1 \right) - e_k \\ &= \left(\tfrac{\gamma\beta}{(1-\gamma)(1-\beta)} - 1 \right) e_k + \tfrac{\gamma}{1-\gamma} \end{aligned}$$

and similarly

$$a_{2k+3} \geq \left(\tfrac{\gamma\beta}{(1-\gamma)(1-\beta)} \right) o_{k+1} + \tfrac{\beta}{1-\beta},$$

from which, with (4·13), a straightforward inductive argument shows that the a_i are positive integers with unbounded odd and even terms.

Now, let the pair of generalised Cantor sets (C, D) be generated from $\left(2^{-1/\alpha}, a\right)$. From Theorem 4·1 we obtain

$$\dim_{\mathrm{LB}} (C \times D) = \dim_{\mathrm{B}} (C \times D) = \dim_{\mathrm{A}} (C \times D) = \alpha$$

and from Lemma 4·3 that $\dim_{\mathrm{A}} C = \dim_{\mathrm{A}} D = \alpha$ as required. Further, from Theorem 4·2,

$$\begin{aligned} \dim_{\mathrm{B}} C &= \alpha \limsup_{k\to\infty} \frac{\sum_{i=1}^{k+1} a_{2j-1}}{\sum_{i=1}^{2k+1} a_i} \\ &= \alpha \limsup_{k\to\infty} \frac{o_{k+1}}{o_{k+1} + e_k} = \alpha \limsup_{k\to\infty} \frac{1}{1 + \frac{e_k}{o_{k+1}}} \end{aligned}$$

and from (4·14) it follows that

$$\frac{1-\beta}{\beta} \frac{e_k}{e_k + 2\frac{1-\beta}{\beta}} \leq \frac{e_k}{o_{k+1}} \leq \frac{1-\beta}{\beta} \frac{e_k}{e_k + \frac{1-\beta}{\beta}},$$

so we conclude that $\dim_{\mathrm{B}} C = \alpha\beta$. A similar argument using (4·15) shows that $\dim_{\mathrm{B}} D = \alpha\gamma$. As in Lemma 4·4 the lower box-counting dimensions are obtained from the chain of dimension inequalities (1·10). $\square$

In conclusion we have introduced equi-homogeneity as a regularity property of a set, which roughly means that there is a uniform bound on the range of the number of balls required in the local covers of the set at each length-scale. We have further shown that a large class of homogeneous Moran sets, including the generalised Cantor sets, are equi-homogeneous, and have used this fact to demonstrate that the class of generalised Cantor sets include natural, elementary examples of sets for which the Assouad dimension product inequality is strict and maximal in the sense that the upper bound

$$\dim_{\mathrm{A}}(C \times D) \leq \dim_{\mathrm{A}} C + \dim_{\mathrm{A}} D \leq 2\dim_{\mathrm{A}}(C \times D)$$

is actually an equality. Further, inside this class of sets are examples that, in addition, have box-counting dimensions with arbitrary values satisfying

$$\dim_{\mathrm{B}} C, \dim_{\mathrm{B}} D \leq \dim_{\mathrm{B}}(C \times D) \leq \dim_{\mathrm{B}} C + \dim_{\mathrm{B}} D,$$

and

$$0 < \dim_{\mathrm{B}} C, \dim_{\mathrm{B}} D < \dim_{\mathrm{B}}(C \times D) < 1.$$

Appendix A. Box-counting dimensions of self-products

The following product dimension equality is interesting, particularly in the light of the parallel result for the Assouad dimension presented here in Lemma 1·6. However, since it falls outside the main scope of this paper we give it in this brief appendix.

Lemma A1. *Let (X, d_X) be a metric space and equip the product space $X \times X$ with a metric satisfying* (1·6)*. For all totally bounded sets $F \subset X$*

$$\dim_{\mathrm{B}}(F \times F) = 2\dim_{\mathrm{B}} F$$

and $$\dim_{\mathrm{LB}}(F \times F) = 2\dim_{\mathrm{LB}} F.$$

Proof. Let $F, G \subset X$ be totally bounded sets. Recall from (1·12) that for all $\delta > 0$

$$\mathcal{N}(F, 4\delta/m_1)\mathcal{N}(G, 4\delta/m_1) \leq \mathcal{N}(F \times G, \delta) \leq \mathcal{N}(F, \delta/m_2)\mathcal{N}(G, \delta/m_2)$$

Consequently,

$$\begin{aligned}\frac{\log \mathcal{N}(F \times G, \delta)}{-\log \delta} &\leq \frac{\log \mathcal{N}(F, \delta/m_2)}{-\log \delta} + \frac{\log \mathcal{N}(G, \delta/m_2)}{-\log \delta} \\ &= \frac{\log \mathcal{N}(F, \delta/m_2)}{-\log(\delta/m_2) + \log(m_2)} + \frac{\log \mathcal{N}(G, \delta/m_2)}{-\log(\delta/m_2) + \log(m_2)}\end{aligned}$$

and

$$\begin{aligned}\frac{\log \mathcal{N}(F \times G, \delta)}{-\log \delta} &\geq \frac{\log \mathcal{N}(F, 4\delta/m_1)}{-\log \delta} + \frac{\log \mathcal{N}(G, 4\delta/m_1)}{-\log \delta} \\ &= \frac{\log \mathcal{N}(F, 4\delta/m_1)}{-\log(4\delta/m_1) + \log(m_1/4)} + \frac{\log \mathcal{N}(G, 4\delta/m_1\delta)}{-\log(4\delta/m_1) + \log(m_1/4)}.\end{aligned}$$

These upper and lower bounds have the same limit superior and the same limit inferior as $\delta \to 0+$, so we obtain

$$\limsup_{\delta \to 0+} \frac{\log \mathcal{N}(F \times G, \delta)}{-\log \delta} = \limsup_{\delta \to 0+} \left(\frac{\log \mathcal{N}(F, \delta)}{-\log \delta} + \frac{\log \mathcal{N}(G, \delta)}{-\log \delta} \right) \tag{A 1}$$

and

$$\liminf_{\delta \to 0+} \frac{\log \mathcal{N}(F \times G, \delta)}{-\log \delta} = \liminf_{\delta \to 0+} \left(\frac{\log \mathcal{N}(F, \delta)}{-\log \delta} + \frac{\log \mathcal{N}(G, \delta)}{-\log \delta} \right). \tag{A 2}$$

Consequently, in the case $F = G$

$$\limsup_{\delta\to 0+} \frac{\log \mathcal{N}(F\times F,\delta)}{-\log\delta} = 2\limsup_{\delta\to 0+}\frac{\log\mathcal{N}(F,\delta)}{-\log\delta}$$

$$\text{and}\qquad \liminf_{\delta\to 0+} \frac{\log \mathcal{N}(F\times F,\delta)}{-\log\delta} = 2\liminf_{\delta\to 0+}\frac{\log\mathcal{N}(F,\delta)}{-\log\delta}.$$

□

We remark that the general box-counting dimension product inequalities follow from (A 1) and (A 2) and the fact that taking limits superior is subadditive whilst taking limits inferior is superadditive.

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