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Topological Fluid Dynamics II

On the regularity of Lagrangian trajectories corresponding to suitable weak solutions of the Navier–Stokes equations

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Abstract

The putative singular set $\mathscr{S}$ in space-time of a suitable weak solution u of the 3D Navier–Stokes equations has box-counting dimension no greater than $5/3$. This allows one to prove that almost all trajectories avoid $\mathscr{S}$. Moreover, for each point x that does not belong to $\mathscr{S}$, one can find a neighbourhood U of x such that the function u is continuous on U and space derivatives of u are bounded on every compact subset of U. It follows that almost all Lagrangian trajectories corresponding to u are C^1 functions of time (Robinson & Sadowski, Nonlinearity 2009). We recall the main idea of the proof, give examples that clarify in what sense the uniqueness of trajectories is considered, and make some comments on how this result might be improved.

Keywords: Navier–Stokes; suitable weak solution; Lagrangian trajectory

1. Introduction

We consider the three-dimensional Navier–Stokes equations for an incompressible fluid:

$$u_t - \Delta u + u \cdot \nabla u + \nabla p = 0, \quad \operatorname{div} u = 0 \tag{1}$$

with an initial condition u_0. The domain of the flow, denoted Ω, is a bounded domain with a smooth boundary and we assume homogeneous boundary conditions: $u = 0$ on $\partial\Omega$.

In what follows by a strong solution of (1) we mean a function $u \in L^\infty(0,T;V^1) \cap L^2(0,T;V^2)$ that satisfies (1) in the distributional sense. Here V^k is the completion of smooth divergence-free functions with compact support in the norm of the Sobolev space H^k. By a weak solution of (1) we mean a function $u \in L^\infty(0,T;V) \cap L^2(0,T;V^1)$ satisfying the Navier–Stokes equations in the distributional sense. The existence of global-in-time weak solutions was established in the famous papers of Leray [1] and Hopf [2].

Strong solutions of the Navier–Stokes equations are unique but their existence has been so far proved only locally in time. On the other hand, weak solutions of (1) exist globally in time but not much is known about their regularity (see

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for example [3]). In particular, they are not known to be unique nor to satisfy the energy equality. Such putative lack of desirable physical properties of weak solutions brings about the question whether weak solutions are sufficiently regular for the construction of the corresponding Lagrangian description of the flow. More precisely, one may ask whether given a weak solution u we can find for each $x_0 \in \Omega$ a corresponding Lagrangian trajectory $\xi_{x_0}(\cdot)$ of a particle that at time $t = 0$ could be found at the point x_0. In the classical formulation we would look for a continuously differentiable function $\xi_{x_0} : [0,T] \to \Omega$ solving the Cauchy problem

$$\begin{cases} \dot{\xi}_{x_0}(t) &= u(\xi_{x_0}(t),t) \\ \xi(0) &= x_0 \end{cases}$$

However, dealing with a weak solution we need to resort to a less restrictive formulation. Therefore, we look for an absolutely continuous function[1] satisfying:

$$\xi_{x_0}(t) = x_0 + \int_0^t u(\xi_{x_0}(s),s)\,\mathrm{d}s. \tag{2}$$

for all $t \in [0,T]$. When $\xi_{x_0}(t)$ is continuously differentiable then this integral equation is equivalent to the original ODE. But when ξ is only an absolutely continuous function satisfying (2) then ξ is differentiable for almost all t and the equation

$$\dot{\xi}_{x_0}(t) = u\left(\xi_{x_0}(t),t\right)$$

does not have to be satisfied for all t but only for almost all. As a result solutions of (2) may represent a trajectory of a particle that rapidly changes the direction of its motion, as shown in Fig 1 below.

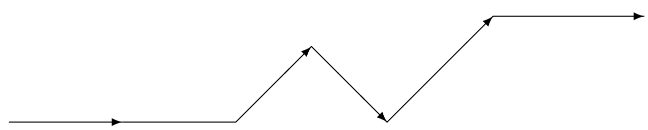

Fig. 1

The existence of absolutely continuous particle trajectories corresponding to a weak solution of the Navier–Stokes equations was established by Foias, Guillopé and Temam. They showed in [4] that for each weak solution u there exists at least one volume-preserving mapping

$$\Phi : \Omega \times [0,T] \to \Omega$$

such that the Lagrangian trajectories are given by

$$\xi_{x_0}(t) = \Phi(x_0,t) \quad \text{for every } x_0 \in \Omega.$$

An important ingredient of the proof is the fact that one can show that any weak solution u is also an element of $L^1(0,T;L^\infty(\bar{\Omega}))$, see [5] or [6]. Since this result can be extended to treat the whole space, one can also obtain a volume-preserving flow map for a weak solution defined on R^3, provided that one replaces the Galerkin procedure used in [4] by an argument based on mollification (cf. Dashti & Robinson [7]).

The result obtained by Foias et al. may be interpreted in the following way: given a weak solution one can choose a family of absolutely continuous trajectories that fit well together to form a measure preserving flow. Is this choice of trajectories unique? The results of DiPerna and Lions [8] guarantee that Φ is indeed the unique volume-preserving mapping up to equality outside a null set of initial data. In other words, if $\Psi\colon \Omega \times [0,T] \to \Omega$ is a volume-preserving mapping such that $\xi_{x_0}(t) = \Psi(x_0,t)$ is a Lagrangian trajectory for each $x_0 \in \Omega$ then

$$\Psi(x_0,t) = \Phi(x_0,t) \quad \forall\, t \in [0,T], \quad \text{for almost every } x_0 \in \Omega.$$

[1]A function $f : [0,T] \to R^n$ is absolutely continuous if for each $\varepsilon > 0$ there is $\delta > 0$ such that if $[\tau_k,t_k]$, $k = 1,2,\ldots,N$ is a family of disjoint subintervals of $[0,T]$ and $\sum_{k=1}^N |t_k - \tau_k| < \delta$ then $\sum_{k=1}^N |f(t_k) - f(\tau_k)| < \varepsilon$.

However, such uniqueness of the mapping Φ does not guarantee almost everywhere uniqueness of trajectories (we illustrate this with a simple example in the next section.) Therefore one can ask further questions about the regularity of Lagrangian trajectories given by the mapping Φ corresponding to a weak solution u:

- Are the trajectories differentiable?
- Do they intersect?
- Are they unique?

If one considers just any weak solution u of the Navier–Stokes equations then to our best knowledge the problems stated above are open. Thus in what follows we will restrict our investigation to a class of global-in-time weak solutions constructed by Caffarelli, Kohn and Nirenberg in [9]. For such weak solutions called 'suitable weak solutions' Caffarelli et. al. proved bounds on the size of the set of space-time singularities. Using these bounds we demonstrate that in fact almost everywhere uniqueness of the trajectories holds, and that these trajectories are not only absolutely continuous but continuously differentiable (with some additional smoothness).

2. Main problem

To answer the questions posed at the end of Introduction we need to work with a concrete representative of a function $u \in L^p$ rather than with the whole class of functions agreeing almost everywhere. The following example makes this clear. Suppose that we want to solve the Cauchy problem:

$$\dot{x}(t) = f(x,t) \qquad x(0) = x_0$$

in the sense defined before: for a given $x \in \Omega$ we look for an absolutely continuous function x satisfying

$$x(t) = x_0 + \int_0^t f(x(s),s)\,\mathrm{d}s$$

for all $t \in [0,T]$.

If $f(x) \equiv 1$ then all trajectories are smooth and they do not intersect at all (Fig 2a).

However, if f is given by

$$f(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

then for all $x < 0$ the Cauchy problem has an infinite number of solutions since every trajectory that reaches the $x = 0$ axis can stay on this axis as long as one wishes (yielding a piecewise smooth and absolutely continuous trajectory), as shown in Fig 2b. Therefore the answers to the questions of the uniqueness and regularity of particular trajectories depend on the choice of a representative of a function $f \in L^p$. This is why a different level of abstraction is needed here than for example in the seminal paper by Di Perna and Lions [8] (see also Cipriano and Cruzeiro [10] or Chemin and Lerner [11]).

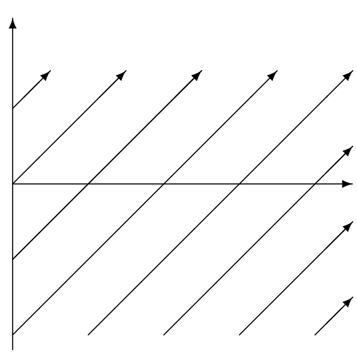

Fig. 2a

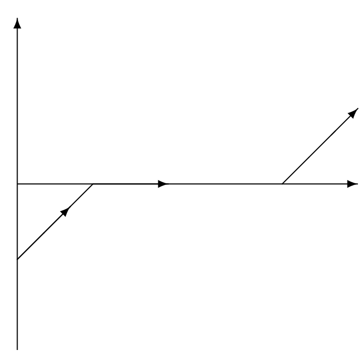

Fig. 2b

In the light of the example just considered it is quite natural to expect that in order to investigate the 'classical' uniqueness of particular trajectories we will need to make use of some 'classical' fine properties of a function u such as continuity or differentiability. However, weak solutions are not known to enjoy such properties on their whole domain. Nevertheless, any weak solution (has a representative that) is a smooth function of x for almost all times except for a small set $\mathscr{T}$ of singular times where the norm $\|u\|_{V^1}$ becomes locally unbounded (this result goes back to Leray). The set $\mathscr{T}$ has box-counting dimension no greater than a half (see for example Robinson & Sadowski [12]), but this fact alone is still not sufficient for our purpose. We need to consider even more regular solutions for which we can show that the set of singular points in space-time is small (in the sense of small box-counting dimension).

Definition. We say that a weak solution u is suitable if the corresponding pressure p belongs to $L^{5/3}((0,T)\times\Omega)$ and u satisfies local energy inequality.

Such weak solutions (global in time) can be constructed though it is not clear whether they can be obtained in a classical way as the limit of Galerkin approximations. Let us now clarify the definitions of singular and regular points in space-time.

Definition. A point $z=(x,t)\in\Omega\times(0,T)$ is called regular if u is continuous in some open neighbourhood of z. A point is called singular if it is not regular. A set of all singular points in space-time we denote by $\mathscr{S}$.

In their famous paper [9] Caffarelli, Kohn and Nirenberg (following the ideas of Scheffer in [13]) proved that a singular set $\mathscr{S}$ in space-time has the Hausdorff dimension no greater than 1 (in fact they used a slightly different definition of singular points; the definition used here is due to Ladyzhenskaya and Seregin [14]; see also Lin [15]). It can also be easy deduced from their proof that the box-counting dimension of $\mathscr{S}$ is no greater than $5/3$. (A better result was obtained by Kukavica [16], but the bound 5/3 is enough for our purpose.) Using this fact and the theorem by Aizenman [17] on avoiding sets of small box-counting dimension one can prove the following theorem (see [18]).

Theorem. Let u be a suitable weak solution of the Navier–Stokes equations (1) with $u_0\in V^{1/2}$. Then almost all Lagrangian trajectories avoid the singular set $\mathscr{S}$ (i.e. they do not intersect $\mathscr{S}$ for all $t\geq 0$).

Let us sketch the proof following the idea from Robinson and Sadowski [19]. For each natural N we divide an interval $[0,T]$ into subintervals $[t_k,t_{k+1}]$ of equal length. Define

$$\delta_k=\int_{t_k}^{t_{k+1}}\|u(s)\|_\infty\,\mathrm{d}s$$

for $k=1,2,...,N$. The numbers δ_k bound from above the maximal distance that a particle can travel between time t_k and t_{k+1}. Let us denote by $\Pi\mathscr{S}$ the projection of the singular set $\mathscr{S}$ onto R^3:

$$\Pi\mathscr{S}=\{x\in R^3:(x,t)\in\mathscr{S}\ \text{ for some }\ t\in[0;T]\}.$$

Moreover for $k=1,2,...,N$ we denote by $\mathscr{O}_{\delta_k}$ the δ_k-neighbourhood of $\Pi\mathscr{S}$:

$$\mathscr{O}_{\delta_k}=\{x\in\Omega:|x-s|\leq\delta_k\ \text{ for some }\ s\in\Pi\mathscr{S}\}.$$

It is easy to see that if at time t_k a particle is not in the set $\mathscr{O}_{\delta_k}$ then the particle avoids the set $\Pi\mathscr{S}$ in the time interval $[t_k,t_{k+1}]$.

We can now use the fact that if a set $X\subset R^n$ has the box-counting dimension d, then for any $d'>d$ there exists a constant $C>0$ such that for all sufficiently small $\varepsilon>0$ the measure of ε-neighbourhood of X can be bounded above by $C\varepsilon^{n-d'}$ (see for example Falconer [20]). Therefore since the projection of $\mathscr{S}$ onto R^3 also has the box-counting dimension no greater than $5/3$ we can take $d'=9/5>5/3$ and deduce that for sufficiently small δ_k the measure of the δ_k-neighbourhood of $\mathscr{S}$ is bounded above by

$$\mu(\mathscr{O}_{\delta_k})\leq C\delta_k^{6/5}.$$

Since the flow is measure preserving the measure of the set of all initial conditions giving rise to a trajectory intersecting the set $\mathscr{S}$ is no greater than

$$\begin{aligned}\sum_{k=1}^{N}\mu(\mathscr{O}_{\delta_k}) &\leq C\sum_{k=1}^{N}\delta_k^{6/5}\\ &\leq C\max_k \delta_k^{1/5}\sum_{k=1}^{N}\delta_k\\ &\leq C\max_k \delta_k^{1/5}\|u\|_{L^1(0,T,L^\infty)}\end{aligned}$$

As N tends to infinity the RHS of the above inequality tends to zero. It follows that almost all trajectories avoid the singular set $\mathscr{S}$ so they consist only of regular points.

It is noteworthy that this portion of the argument shows, in fact, that any set with box-counting dimension strictly smaller than 2 would be avoided by almost every trajectory. This suggests strongly that one might be able to limit further the set of initial conditions that intersect $\mathscr{S}$, by restricting their dimension (or Hausdorff measure).

We can now use standard results about the regularity of u in a neighbourhood of regular points (see for example Serrin [21] and Skalak & Kucera [22]): for each regular point $z_0 = (x_0, t_0)$ we can find $r > 0$ such that

- $u(\cdot, t)$ is smooth on
$$B_r(x_0) = \{x \in \Omega : |x - x_0| \leq r\}$$
for all t such that $|t - t_0| < r^2$
- for every multi-index γ the derivative $D_x^\gamma u$ is continuous in
$$B_r(z) = \{z \in \Omega \times [0,T] : |z - z_0| \leq r\}.$$

It follows now that the trajectories are C^1 functions of time. In fact one can show that in a neighbourhood of a regular point $D_x^\gamma u \in C^{0,\beta}$ for all $0 < \beta < 1/2$. Better results are known for the whole space, in which case $\beta \in (0,1)$.

It is not clear whether these results can be improved. In particular, it is an open problem whether or not almost all Lagrangian trajectories are C^2 functions of time.

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References

[1] Leray, J. Sur le mouvement d'un liquide visqueux emplissant l'espace. *Acta Math.* 63 (1934), pp. 193–248.

[2] Hopf, E. Über die Anfangswertaufgabe dür die hydrodynamischen Grundgleichungen. *Math. Nachr.* 4 (1951): 213–231.

[3] Constantin, P. and C. Foias. *Navier–Stokes equations*, University of Chicago Press, Chicago; 1988.

[4] Foias, C., C. Guillopé, and R. Temam. Lagrangian representation of a flow. *J. Diff. Eq.* 57 (1985): 440–449.

[5] Foias, C., C. Guillopé, and R. Temam. New a priori estimates for Navier-Stokes equations in dimension 3. *Comm. Partial Differential Equations* 6 (1981): 329–359.

[6] Foias, C., O. Manley, R. Rosa, and R. Temam. *Navier–Stokes equations and turbulence*. Cambridge University Press, Cambridge; 2001.

[7] Dashti, M., and J.C. Robinson. A simple proof of uniqueness of the particle trajectories for solutions of the Navier–Stokes equations. *Nonlinearity* 22 (2009): 735–746.

[8] DiPerna, R.J., and P.L. Lions. Ordinary differential equations, transport theory and Sobolev spaces. *Invent. math.* 98 (1989): 511–547.

[9] Caffarelli, L., R. Kohn, and L. Nirenberg. Partial regularity of suitable weak solutions of the Navier-Stokes equations. *Comm. Pure Appl. Math.* 35 (1982): 771–831.

[10] Cipriano, F., and A.B. Cruzeiro. Flows associated with irregular R^d–vector fields. *J. Diff. Eq.* 219 (2005): 183–201.

[11] Chemin, J.Y., and N. Lerner. Flot de champs de vecteurs non lipschitziens et équations de Navier–Stokes. *J. Diff. Eq.* 121 (1995): 314–328.

[12] Robinson, J.C., and W. Sadowski. Decay of weak solutions and the singular set of the three-dimensional Navier-Stokes equations. *Nonlinearity* 20 (2007): 1185–1191.

[13] Scheffer, V. The Navier-Stokes equations on a bounded domain. *Comm. Math. Phys.* 73 (1980): 1–42.

[14] Ladyzhenskaya, O., and G. Seregin. On partial regularity of suitable weak solutions to the three-dimensional Navier-Stokes equations. *J. Math. Fluid Mech.* 1 (1999): 356–387.

[15] Lin, F. A New Proof of the Caffarelli-Kohn-Nirenberg Theorem. *Comm. Pure Appl. Math* 51 (1998): 241–257.

[16] Kukavica, I. The fractal dimension of the singular set for solutions of the Navier–Stokes system. In preparation.

[17] Aizenman, M. A sufficient condition for the avoidance of sets by measure preserving flows in R^N. *Duke Math. J.* 45 (1978): 809-812.

[18] Robinson, J.C., and W. Sadowski. Almost everywhere uniqueness of Lagrangian trajectories for suitable weak solutions of the three-dimensional Navier–Stokes equations. *Nonlinearity.*, 2009.

[19] Robinson, J.C., and W. Sadowski. A criterion for uniqueness of Lagrangian trajectories for weak solutions of the 3D Navier-Stokes equations. *Comm. Math. Phys.*, 2009.

[20] Falconer, K. *Fractal Geometry: Mathematical Foundations and Applications.* (2nd edition) Chichester: John Wiley & Sons, 2003.

[21] Serrin, J. On the interior regulariy of weak solutions of the Navier-Stokes equations. *Arch. Rat. Mech. Anal.* 9 (1962): 187–191.

[22] Skalak, Z., and P. Kucera. Regularity of pressure in the neighbourhood of regular points of weak solutions of the Navier–Stokes equations” *Applications of Mathematics* 48 (2003)