ASYMPTOTIC EXPANSIONS IN THE PROBLEM
OF THE LENGTH OF THE LONGEST HEAD-RUN
FOR MARKOV CHAIN WITH TWO STATES

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(abridged version)

Let \{\xi_i, i \geq 0\} be a homogeneous Markov chain with
states \{0; 1\}, transition probabilities \(p_{11} = \alpha, p_{00} = \beta\),
\(0 < \alpha < 1, \beta < 1\), and initial distribution \(\mathbb{P}(\xi_0 = 1) = p\). We set

\[ \eta_n = \max\{k \leq n: \max_{0 \leq i \leq n-k} 1_{\{\xi_{i+1} = \ldots = \xi_{i+k} = 1\}} = 1\} \quad (0) \]

Random variable \(\eta_n\) is known in literature as the length of
the longest head-run.

V.I. Goncharov [1] proved that in the case of Bernoulli
scheme we have: for any \(j \in \mathbb{Z}\)

\[ \mathbb{P}(\eta_n - (\log n) - j) = \exp(-\log^{\frac{1}{\alpha}} - \log n) + o(1) \quad (n + \infty), \]

where \(\log\) is to base \(1/\alpha\), \([x]\) is the integer part of \(x\),
\((x) = x - [x]\).

Analogous results for more general situations were obtained in
[2-7]. Assertions of LIL type were found in
[4,5,8-13]. Moivre [14] seems to be the first who suggested
to study the distribution of the length of the longest head-run.

The purpose of this article is to find asymptotic
expansions in the limit theorem for the distribution of r.v.
\(\eta_n\).
§ 1. Main theorem

Let \( r = (1 - \alpha)(1 - \beta)/\alpha(2 - \alpha - \beta) \) and

\[
Y_i(k, \phi) = \phi^i \sum_{j=0}^i \sum_{d=0}^j T_{j-d} \sum_{\mu=0}^d Q_{\mu, d} \sum_{h \in \mathbb{N}} \sum_{i-j \geq 0} \left< \frac{-i-j}{c_i} \right>^{-1} \sum_{\nu=0}^{\lambda=0} \left< \frac{-1}{c_i} \right>^{d+\mu+\lambda} \phi^{-i} \quad (i \geq 0)
\]

where

\[
i^{(d)} = i(i-1) \ldots (i-d+1) \quad (d \geq 1),
\]

\[
i^{(e)} = 1, \quad i^{(-d)} = 0
\]

functions \( T, Q, h \) are defined by formulae (2.7), (2.16), (2.12).

Note that \( Y_i(k, \phi) \), as a function of the first argument, is a polynomial of degree \( i \); it is a polynomial of degree \( 2i \) as a function of the second argument.

Theorem 1. For any \( m \geq 1 \) there exists constant \( C_m = C(m, \alpha, \beta, \rho) \) such that for \( n > C_m \) there holds

\[
\sup_{-\infty < j < +\infty} |P(\eta_n - \{\log n\} < j) - e^{-\phi_{n_j}^{m-1}} \sum_{i=0}^{\infty} Y_i(k_{n_j}, \phi_{n_j})| \leq C_m n^{-1} \ln n^{3m}
\]

(1.3)

where \( k_{n_j} = j + \{\log n\} \), \( \phi_{n_j} = \gamma \alpha^{j - \{\log n\}} \).

Corollary. For \( n \to \infty \) we have

\[
\sup_{-\infty < j < +\infty} |P(\eta_n - \{\log n\} < j) -
\]


\[- \varphi_n^j \left( 1 + \varphi_n^j (1 - \varphi_n^j)^{-1} \log n \right) = O(n^{-1}) \quad (1.4)\]

Note that the first and the second terms of the expansion both do not depend on the initial distribution of the chain.

\[\S 2. \text{Some auxiliary results}\]

In the sequel letters \( C, c \) (with indexes or without) denote constants which depend on \( m \) and chain parameters only.

**Theorem 2.** There exist constants \( q < 1 \) and \( C < \infty \) such that

\[\sup_{k \in C} \left| \mathbb{P} \left( \eta_n < k \right) - \mathcal{A}(t_o) l_o^{-n-1} \right| \leq C q^n \quad (2.1)\]

where

\[A(t) = V(t)/U(t),\]

\[V(t) = V(t, k) = 1 - (\alpha + \beta - 1) t - \]

\[- (\alpha + (1 - p)(1 - \beta)) \alpha^{k-1} t^k + p(\alpha + \beta - 1) \alpha^{k-1} t^k + 1,\]

\[U(t) = U(t, k) = W(t) + (1 - \alpha)(1 - \beta) \alpha^{k-1} t^k + 1,\]

\[W(t) = (1 - t)(1 - (\alpha + \beta - 1) t),\]

\[t_o = t_o(k) \text{ is a root of } U(t, k) \text{ with minimal modulus.}\]

In the case of Bernoulli \( B(\alpha) \) scheme we have \( q = \alpha \) and \( C = (2 + \alpha(1 + \alpha))/(1 - \alpha)(1 - \alpha^2) \).

**Lemma 1.** For \( k \geq 1 \) we have

\[F(k, t) = \sum_{n=0}^{\infty} \mathbb{P} \left( \eta_n < k \right) t^n = V(t)/U(t) \quad (2.2)\]

where \( \eta_0 = 0 \).

Denote \( \kappa = (\alpha + \beta - 1)/(2 - \alpha - \beta) \), \( \delta = 1 - p/\gamma - (1 - p)/(1 - \alpha) \),

\[\rho = (\alpha + \beta - 1)/(1 - \alpha)(1 - \beta), \quad H_i = \theta \quad (i \neq 0),\]

\[H_i = H_i(k) = \quad (2.6)\]
\[
-4 - \\
= 2^{-i} i^{2j} \prod_{j=0}^{i+1} \frac{(k+2 \omega)^{i-2j} - (4(k-1) \omega)^{i-2j}}{(k+2 \omega)^{2j} - 4(k-1) \omega^{2j}} \quad (i \geq 0)
\]

We put
\[
T_i = T_i(k) = \sum_{j=0}^{3} q_j H_{i-j},
\]
where \( q_o = 1, \ q_1 = 6 - \omega, \ q_2 = \omega - \omega \omega \), \( q_3 = -\omega \omega \).

Lemma 2. For all \( k \) large enough we have
\[
A(1+u) = \sum_{i=0}^{\infty} T_i u^i
\]
where \( u = u(k) = t_o(k) - 1 \).

Note that
\[
|H_i(k)| \leq (k+2 \omega)^i, \quad |T_i(k)| \leq C k^i
\]
(2.9)

We define polynomials \( P_i(\cdot) \) by the equalities \( P_0 = 0 \),
\[
P_m = P_m(k) = \sum_{j=1}^{m} G_j(k) b_{m-j,j}(k) \quad (k \geq m \geq 1),
\]
where \( G_j(k) = \sum_{j=0}^{i} c_i^j k^{j-1} \) and
\[
b_{l,j} = b_{l,j}(k) = \sum_{i_1 + \ldots + i_j = l} P_{i_1} \ldots P_{i_j} \quad (i \geq 0, j \geq 1)
\]

Let \( v = v(k) = r \omega^k \).

Lemma 3. For any \( m \geq 1 \) there exist constants \( c_m, k_m \) such that for \( k \geq k_m \) we have
\[
|u/v - \sum_{i=0}^{m-1} P_i v^i| \leq c_m (kv)^m
\]
(2.10)

We introduce functions \( \tilde{P}_i, i \geq 0 \) by the equalities
\[
\tilde{P}_i = P_i \quad (0 \leq i \leq m),
\]
\[
\tilde{P}_i v^m = u/v - \sum_{i=0}^{m-1} P_i v^i
\]
We put also
\[ b_{l,j,m} = \sum_{i_1 + \ldots + i_j = l} \tilde{p}_{i_1} \tilde{p}_{i_2} \ldots \tilde{p}_{i_j} \quad \forall m, l, j \leq m \]

Note that
\[ b_{l,j} = \sum_{\nu=1}^{l} j(\nu) h(\nu, l)/\nu! \quad (l \geq 1, j \geq 1) \quad (2.11) \]

where \( h(\nu, l) = h(\nu, l, k) \) is a polynomial (as function of \( k \)) defined by the equalities \( h(0, 0) = 1, \ h(0, l) = 0 \ (l \geq 1) \),

\[ h(\nu, l) = h(\nu, l, k) = \sum_{1 \leq M \leq \nu} \sum_{(y, z) \in A(\nu, l, M)} (\nu!/z!) \cdot \left( \prod_{j=1}^{M} P_{y_j}(k) \right)^{z_j} \quad (l \geq 1, \nu \geq 1) \quad (2.12) \]

Here \( \nu' = \min(\nu; \sqrt{2l}) \); \( y = \{y_1, \ldots, y_M\} \); \( z = \{z_1, \ldots, z_M\} \);
\( z! = z_1! \ldots z_M! \);
\( A(\nu, l, M) = \{(y, z): 1 \leq y_1 < \ldots < y_M; \min_{i} z_i \geq 1; \sum_{i=1}^{M} z_i = \nu; \sum_{i=1}^{M} y_i z_i = l\} \)

Similarly
\[ b_{l,j,m} = \sum_{\nu \geq \delta_{\nu}}^{l} j(\nu) h_{m}(\nu, l)/\nu! \quad , \quad (2.13) \]

where \( h_m(0, 0) = 1 \), definition of \( h_m(\nu, l) \) differs from that of \( h(\nu, l) \) by using \( \tilde{p}_{i} \) instead of \( p_{i} \) and \( A(\nu, l, M, m) \) instead of \( A(\nu, l, M) \), where
\[ A(\nu, l, M, m) = \{(y, z) \in A(\nu, l, M): \max_{1 \leq i \leq M} y_i \leq m\} \]

Note that \( h_{m}(\nu, l) = h(\nu, l) \) as \( \nu < m \) and
\[ |h_{m}(\nu, l, k)| \leq 2^{m} \nu^{m}(c_{m}k)^{l} \quad (2.14) \]
\[ |b_{l,j,m}(k)| \leq 2^{m(m+1)}j^{l}(c_{m}k)^{l} \quad (2.15) \]
Lemma 4. Let $S_d(i) = \sum_{r=0}^{i} r^{(d)}$. Then for $d \geq 0$ we have

$$S_d(i) = (i+1)^{d+1}/(d+1) = i^{(d)} + i^{(d+1)}/(d+1)$$

Corollary.

$$S_d(i) = \sum_{j=0}^{i} S_{d-1}(j-1) \quad (d \geq 1)$$

$$(i+1)S_d(i-1) = (d+2)(d+1)^{-1}S_{d+1}(i) \quad (i \geq 1)$$

$$S_d(i+1) = S_d(i) + dS_{d-1}(i) \quad (d \geq 1)$$

Lemma 5. Let coefficients $r_j(i)$ be defined by the equality

$$(n+1)^{i} \cdots (n+1)^{0} = \sum_{j=0}^{i} r_j(i)n^{i-j},$$

and let

$$Q_{0,d} = 1, \quad Q_{j,d} = \sum_{1<l_1<l_1+1<\ldots<l_j<d+j} l_1 l_2 \cdots l_j^{-1}$$

for $1 \leq j < d$. If $d \geq 1$ then we have

$$r_d(i) = \sum_{j=0}^{d-1} Q_{j,d}S_{j+d}(i)$$

There follows from lemmas 4, 5 that

$$r_d(i) = \sum_{j=0}^{d} Q_{j,d}S_{j+d}(i+1)^{j+d} \quad (d \geq 0) \quad (2.16)$$

where $Q_{0,0} = 1, \quad Q_{0,d} = 0 \quad (d \geq 1), \quad Q_{j,d} = (j+d)^{-1}Q_{j-1,d} \quad (1 \leq j \leq d)$. 

Lemma 6. Let $\alpha, \nu \in \mathbb{Z}; \nu \geq 0$. Then

$$i^{(\nu)} = \sum_{\lambda=0}^{\nu} a(\nu-\lambda)\alpha(\lambda)(i-\alpha)(\lambda) \quad (2.17)$$

§ 3. Proof of the main result.
We define \( Y_{i,m} = Y_{i,m}(k,\phi) \) by using \( h_{m}(\nu,l) \) instead of \( h(\nu,l) \) in formula (1.1). In the sequel \( \phi = n\nu \).

Lemma 7. For all \( k \) large enough we have
\[
A_{<\phi,\tilde{t}_{o}>}^{-n-1} = e^{-\phi \sum_{i=0}^{\infty} y_{i,m}}
\] (3.1)

Lemma 8. Let \( \psi = \max(1;\phi) \). Then
\[
|y_{i,m}| \leq (c\psi^{2}ln n)^{i}
\] (3.6)

Let \( k(n) = \log n - \log \ln n^{m} \) (log is to base \( 1/\alpha \)).

Lemma 9. If \( m>1 \), then for all \( n \) large enough we have
\[
\sup_{k \in \mathbb{Z}} \left| \mathbb{P}(\eta_{k} - \sum_{i=0}^{m-1} e^{-\phi \sum_{i=0}^{\infty} y_{i}} \leq Cq^{n} + \right.
\]
\[
+ 2 \sup_{k \leq k(n)} \sum_{i=0}^{\infty} e^{-\phi \sum_{i=0}^{\infty} y_{i}} + \sup_{k \geq k(n)} \sum_{i=0}^{m} e^{-\phi \sum_{i=0}^{\infty} y_{i,m}}
\]
where \( q<1 \).

Let
\[
\tilde{\eta}_{n} = \max\{k \leq n: \max_{0 \leq i \leq n-k} \{\xi_{i}=\ldots=\xi_{i+k-1}=1\} = 1\}
\]

It is easy to see that assertion (1.3) holds if we define \( Y_{i} \) using \( \tilde{T}_{i} \) instead of \( T_{i} \), where \( \tilde{T}_{i} = \sum_{j=0}^{k} q_{j} H_{i-j} \), \( q_{0}=1 \), \( q_{1} = 1 - \kappa + \beta - p/(1-\alpha)(1-\beta) \), \( q_{2} = (1-\kappa)p - \kappa + \kappa \hat{\rho}/(1-\alpha)(1-\beta) \), \( q_{3} = -\kappa p \), \( \hat{\rho} = \alpha(p+\beta-1)/(1-\alpha)(1-\beta) \).

§ 4. Remark on the rate of convergence.
§ 4. Remark on the rate of convergence.

Let \( \{X_n; n \geq 1\} \) be a Markov chain with state space \( S = \{0, 1, \ldots, m\} \), transition probabilities \( p_{ij} \) and initial distribution \( \bar{p} \). We define r.v. \( \eta_n \) by equality \( (0) \), where \( \xi_i = 1 \{X_i \leq A\} \), \( A = \{1, \ldots, m\} \).

Let \( \lambda \) be a maximal eigenvalue of the matrix \( U = \|p_{ij}\|_{ij \in A} \). We introduce r.v. \( \xi \) with distribution

\[
P(\xi = i) = p_{00}, \quad P(\xi = i) = \bar{p}_{0A} \bar{p}_{iA}^{i-2} \bar{p}_{0A} \quad (i \geq 2),
\]

where \( \bar{p}_{0A} = \|p_{0j}\|_{j \in A} \), \( \bar{p}_{iA} = \|p_{i0}\|_{i \in A} \). We suppose that there is only one class \( C \) of essential states, which has no cyclic subclasses; \( A \cap C \neq \emptyset \); \( 0 < \lambda < 1 \); corresponding right eigenvector \( \bar{z} \) of matrix \( U \) is positive: \( z_j > 0 \) \( (1 \leq j \leq m) \).

Theorem 3. Let \( a(k) = 1P(\xi > k) \) and

\[
\Delta(C_n, k) = | 1P(\eta_n < k) - \exp(-na(k)/\lambda(k)) | \quad (4.1)
\]

Then \( \sup_{1 \leq k \leq n} \Delta(C_n, k) = O(n(\ln n)^r) \) as \( n \to \infty \).

Let \( \tau_i \) be the i-th zero in the sequence \( \{X_n; n \geq 1\} \) and let \( \xi_i = \tau_i - \tau_i - 1 \). Then

\[
\eta_n = \max \{ n - \nu(n); \max_{1 \leq i \leq \nu(n)} \xi_i \} \quad (4.2)
\]

where \( \nu(n) = \max \{i: \tau_i \leq n\} \). The proof is based on the fact that \( 1P(\eta_n < k) \propto \mathcal{M}(1 - \alpha(k))^\nu(n, k) \), where \( \nu(n, k) = \max \{r: \sum_{j=1}^{r} \xi_j(k) \leq n\} \), r.v.'s \( \xi_j(k) \) are independent and have the distribution \( 1P(\xi_j(k) = i) = 1P(\xi_j = i | \xi_j \leq k) \).
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